

# PROBABILISTIC CALL BY PUSH VALUE

THOMAS EHRHARD AND CHRISTINE TASSON

CNRS, IRIF, UMR 8243, Univ Paris Diderot,, Sorbonne Paris Cité, F-75205 Paris, France  
*e-mail address:* thomas.ehrhard@pps.univ-paris-diderot.fr

CNRS, IRIF, UMR 8243, Univ Paris Diderot,, Sorbonne Paris Cité, F-75205 Paris, France  
*e-mail address:* christine.tasson@pps.univ-paris-diderot.fr

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**ABSTRACT.** We introduce a probabilistic extension of Levy’s Call-By-Push-Value. This extension consists simply in adding a “flipping coin” boolean closed atomic expression. This language can be understood as a major generalization of Scott’s PCF encompassing both call-by-name and call-by-value and featuring recursive (possibly lazy) data types. We interpret the language in the previously introduced denotational model of probabilistic coherence spaces, a categorical model of full classical Linear Logic, interpreting data types as coalgebras for the resource comonad. We prove adequacy and full abstraction, generalizing earlier results to a much more realistic and powerful programming language.

## 1. INTRODUCTION

*Call-by-Push-Value* [16] is a class of functional languages generalizing the lambda-calculus in several directions. From the point of view of Linear Logic we understand it as a half-polarized system bearing some similarities with e.g. classical Parigot’s lambda-mu-calculus, this is why we call it  $\Lambda_{\text{HP}}$ . The main idea of Laurent and Regnier interpretation of call-by-name lambda-mu in Linear Logic [14] (following actually [11]) is that all types of the minimal fragment of the propositional calculus (with  $\Rightarrow$  as unique connective) are interpreted as *negative* types of Linear Logic which are therefore naturally equipped with structural morphisms: technically speaking, these types are algebras of the  $?$ -monad of Linear Logic. This additional structure of negative types allows to perform logical structural rules on the *right side* of typing judgments even if these formulas are not necessarily of shape  $?\sigma$ , and this is the key towards giving a computational content to classical logical rules, generalizing the fundamental discovery of Griffin on typing call/cc with Peirce Law [12].

From our point of view, the basic idea of  $\Lambda_{\text{HP}}$  is quite similar, though somehow dual and used in a less systematic way: data types are interpreted as *positive* types of Linear Logic equipped therefore with structural morphisms (as linear duals of negative formulas, they are coalgebras of the  $!$ -comonad) and admit therefore structural rules on the *left side* of typing judgment even if they are not of shape  $!\sigma$ . This means that a function defined on a data type can have a *linear function type* even if it uses its argument in a non-linear way:

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this non-linearity is automatically implemented by means of the structural morphisms the positive data type is equipped with.

The basic positive type in Linear Logic is  $!\sigma$  (where  $\sigma$  is any type): it is the very idea of Girard’s call-by-name translation of the lambda-calculus into Linear Logic to represent the implication type  $\sigma \Rightarrow \tau$  by means of the decomposition  $!\sigma \multimap \tau$ . The new idea of  $\Lambda_{\text{HP}}$  is to generalize this use of the linear implication to any type construction of shape  $\varphi \multimap \tau$  where  $\varphi$  is a positive type, without imposing any linearity restriction on the usage of the argument of type  $\varphi$  used by a function of type  $\varphi \multimap \tau$ . This non-symmetrical restriction in the use of the linear implication motivates our description of  $\Lambda_{\text{HP}}$  as a “half-polarized” system: in a fully polarized system as Laurent’s *Polarized Linear Logic* LLP [14], one would also require the type  $\sigma$  to be negative in  $\varphi \multimap \sigma$  (the last system presented in [3] implements this idea) and the resulting formalism would host classical computational primitives such as call/cc as well. The price to pay, as illustrated in [1], is a less direct access to data types: it is impossible to give a function from integers to integers the expected type  $\iota \multimap \iota$  (where  $\iota$  is the type of flat natural numbers satisfying  $\iota = \mathbf{1} \oplus \iota$ ), the simplest type one can give to such a term is  $\iota \multimap ?\iota$  which complicates its denotational interpretation<sup>1</sup>.

Not being polarized on the right side of implications,  $\Lambda_{\text{HP}}$  remains “intuitionistic” just as standard functional programming languages whose paradigmatic example is PCF. So what is the benefit of this special status given to positive formulas considered as “data types”? There are several answers to this question.

- First, and most importantly, it gives a *call-by-value access* to data types: when translating PCF into Linear Logic, the simplest type for a function from integers to integers is  $!\iota \multimap \iota$ . This means that arguments of type  $\iota$  are used in a call-by-name way: such arguments are evaluated again each time they are used. This can of course be quite inefficient. It is also simply *wrong* if we extend our language with a random integer generator since in that case each evaluation of such an argument can lead to a different value: in PCF there is no way to keep memory of the value obtained for one evaluation of such a parameter and probabilistic programming is therefore impossible. In  $\Lambda_{\text{HP}}$  data types such as  $\iota$  can be accessed in call-by-value, meaning that they are evaluated once and that the resulting value is kept for further computation: this is typically the behavior of a function of type  $\iota \multimap \iota$ . This is not compulsory however and an explicit  $!$  type constructor still allows to define functions of type  $!\iota \multimap \iota$  in  $\Lambda_{\text{HP}}$ , with the usual PCF behavior.
- Positive types being closed under positive Linear Logic connectives (direct sums and tensor product) and under “least fix-point” constructions, it is natural to allow corresponding constructions of positive types in  $\Lambda_{\text{HP}}$  as well, leading to a language with rich data type constructions (various kinds of trees, streams etc are freely available) and can be accessed in call-by-value as explained above for integers. From this data types point of view, the  $!$  Linear Logic connective corresponds to the type of *suspensions* which are boxes (in the usual Linear Logic sense) containing unevaluated pieces of program.

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<sup>1</sup>One can also consider  $?$  as the computational monad of linear continuations and use a translation from direct style into monadic style (which, for this monad, is just a version of the familiar CPS translation). This is just a matter of presentation and of syntactic sugar and does not change the denotational interpretation in the kind of concrete models of Linear Logic we have in mind such as the relational model, the coherence space model etc.

- Since the Linear Logic type constructors  $\multimap$  and  $!$  are available in  $\Lambda_{\text{HP}}$  (with the restriction explained above on the use of  $\multimap$  that the left side type must be positive), one can represent in  $\Lambda_{\text{HP}}$  both Girard’s translations from lambda-calculus into Linear Logic introduced in [10]: the usual one which is call-by-name and the “boring one” which is call-by-value. So in some sense  $\Lambda_{\text{HP}}$  allows to freely combine these two styles of functional program evaluation.

Concretely, in  $\Lambda_{\text{HP}}$ , a term of positive type can be a *value*, and then it is discardable and duplicable and, accordingly, its denotational interpretation is a morphism of coalgebras: values are particular terms whose interpretation is easily checked to be such a morphism, which doesn’t preclude other terms of positive type to have the same property of course, in particular terms reducing to values! Being a value is a property which can be decided in time at most the size of the term and values are freely duplicable and discardable. The “ $\beta$ -rules” of the calculus (the standard  $\beta$ -reduction as well as the similar reduction rules associated with tensor product and direct sum) are subject to restrictions on certain subterms of redexes to be values (because they must be duplicable and discardable) and these restrictions make sense thanks to this strong decidability property of being a value.

**Probabilities in  $\Lambda_{\text{HP}}$ .** Because of the possibility offered by  $\Lambda_{\text{HP}}$  of handling values in a call-by-value manner, this language is particularly suitable for probabilistic functional programming. Contrarily to the common monadic viewpoint on effects, we consider an extension of the language where probabilistic choice is a primitive  $\text{coin}(p)$  of type  $\mathbf{1} \oplus \mathbf{1}$  (the type of booleans)<sup>2</sup> parameterized by  $p \in [0, 1] \cap \mathbb{Q}$  which is the probability of getting **t** (and  $1-p$  is the probability of getting **f**). So our probabilistic extension is in direct style, but, more importantly, the denotational semantics we consider is itself in “direct style” and does not rely on any auxiliary computational monad of probability distributions, measures, random variables or whatsoever (see [13] for the difficulties related with the monadic approach to probabilistic computations).

On the contrary, we interpret our language in the model of *probabilistic coherence spaces* [2] that we already used for providing a fully abstract semantics for probabilistic PCF [8]. A probabilistic coherence space  $X$  is given by an at most countable set  $|X|$  (the web of  $X$ ) and a set  $\text{PX}$  of  $|X|$ -indexed families of non-negative real numbers, to be considered as some kind of “generalized probability distributions”. This set of families of real numbers is subject to a closure property implementing a simple intuition of probabilistic observations. Probabilistic coherence spaces are a model of classical Linear Logic and can be seen as  $\omega$ -continuous domains equipped with an operation of convex linear combination, and the linear morphisms of this model are exactly the Scott continuous functions commuting with these convex linear combinations.

As shown in [2] probabilistic coherence spaces have all the required completeness properties for interpreting recursive type definitions (that we used in [6] for interpreting the pure lambda-calculus) and so we are able to associate a probabilistic coherence space with all types of  $\Lambda_{\text{HP}}$ .

In this model the type  $\mathbf{1} \oplus \mathbf{1}$  is interpreted as the set of sub-probability distributions on  $\{\mathbf{t}, \mathbf{f}\}$  so that we have a straightforward interpretation of  $\text{coin}(p)$ . Similarly the type of flat integers  $\iota$  is interpreted as a probabilistic coherence space  $\mathbf{N}$  such that  $|\mathbf{N}| = \mathbb{N}$  and  $\text{PN}$  is the set of all probability distributions on the natural numbers. Given probabilistic spaces

<sup>2</sup>An not of type  $T(\mathbf{1} \oplus \mathbf{1})$  where  $T$  would be a computational monad of probabilistic computations.

$X$  and  $Y$ , the space  $X \multimap Y$  has  $|X| \times |Y|$  as web and  $P(X \multimap Y)$  is the set of all  $|X| \times |Y|$  matrices which, when applied to an element of  $PX$  gives an element of  $PY$ . The web of the space  $!X$  is the set of all finite multisets of elements of  $|X|$  so that an element of  $!X \multimap Y$  can be considered as a power series on as many variables as there are elements in  $|X|$  (the composition law associated with the Kleisli category of the  $!$ -comonad is compatible with this interpretation of morphisms as power series).

From a syntactic point of view, the only values of  $\mathbf{1} \oplus \mathbf{1}$  are  $\mathbf{t}$  and  $\mathbf{f}$ , so  $\text{coin}(p)$  is not a value. Therefore we cannot reduce  $\langle \lambda x^{\mathbf{1} \oplus \mathbf{1}} M \rangle \text{coin}(p)$  to  $M[\text{coin}(p)/x]$  and this is a good thing since then we would face the problem that the boolean values of the various occurrences of  $\text{coin}(p)$  might be different. We have first to reduce  $\text{coin}(p)$  to a value, and the reduction rules of our probabilistic  $\Lambda_{\text{HP}}$  stipulate that  $\text{coin}(p)$  reduces to  $\mathbf{t}$  with probability  $p$  and to  $\mathbf{f}$  with probability  $1 - p$  (in accordance with the kind of operational semantics that we considered in our earlier work on this topic, starting with [2]). So  $\langle \lambda x^{\mathbf{1} \oplus \mathbf{1}} M \rangle \text{coin}(p)$  reduces to  $M[\mathbf{t}/x]$  with probability  $p$  and to  $M[\mathbf{f}/x]$  with probability  $1 - p$ , which is perfectly compatible with the intuition that in  $\Lambda_{\text{HP}}$  application is a linear operation (and that implication is linear: the type of  $\lambda x^{\mathbf{1} \oplus \mathbf{1}} M$  is  $(\mathbf{1} \oplus \mathbf{1}) \multimap \sigma$  for some type  $\sigma$ ): in this operational semantics as well as in the denotational semantics outlined above, linearity corresponds to commutation with (probabilistic) convex linear combinations.

**Contents.** The results presented in this paper illustrate the tight connection between the syntactic and the denotational intuitions underpinning our understanding of this calculus. We prove first an Adequacy Theorem whose statement is extremely simple: given a closed term  $M$  of type  $\mathbf{1}$  (which has exactly one value  $()$ ), the denotational semantics of  $M$ , which is an element of  $[0, 1]$ , coincides with its probability to reduce to  $()$  (such a term can only diverge or reduce to  $()$ ). In spite of its simple statement the proof of this result requires some efforts mainly because of the presence of unrestricted recursive types in  $\Lambda_{\text{HP}}$ . The method used in the proof relies on an idea of Pitts [19] and is described in the introduction of Section 3.4.

Then we prove Full Abstraction in Section 4 adapting the technique used in [6] to the present  $\Lambda_{\text{HP}}$  setting. The basic idea consists in associating, with any element  $a$  of the web of the probabilistic coherence space  $[\sigma]$  interpreting the type  $\sigma$ , a term  $a^-$  of type  $!\sigma \multimap !\iota \multimap \mathbf{1}$  such that, given two elements  $w$  and  $w'$  of  $P[\sigma]$  such that  $w_a \neq w'_a$ , the elements  $[a^-]w$  and  $[a^-](w')$  of  $P(!\iota \multimap \mathbf{1})$  are different power series depending on a finite number  $n$  of parameters (this number  $n$  depends actually only on  $a$ ) so that we can find a rational number valued sub-probability distribution for these parameters where these power series take different values in  $[0, 1]$ . Applying this to the case where  $w$  and  $w'$  are the interpretations of two closed terms  $M$  and  $M'$  of type  $\sigma$ , we obtain, by combining  $a^-$  with the rational sub-probability distribution which can be represented in the syntax using  $\text{coin}(p)$  for various values of  $p$ , a  $\Lambda_{\text{HP}}$  closed term  $C$  of type  $!\sigma \multimap \mathbf{1}$  such that the probability of convergence of  $\langle C \rangle M$  and  $\langle C \rangle (M')$  are different (by adequacy). This proves that if two (closed) terms are operationally equivalent then they have the same semantics in probabilistic coherence spaces, that is, equational full abstraction.

**Further developments.** These results are similar to the ones reported in [9] but are actually different, and there is no clear logical connection between them, because the languages are quite different, and therefore, the observation contexts also. And this even in spite of

the fact that PCF can be faithfully encoded in  $\Lambda_{\text{HP}}$ . This seems to show that the semantic framework for probabilistic functional programming offered by probabilistic coherence spaces is very robust and deserves further investigations. One major outcome of the present work is a natural extension of probabilistic computation to rich data-types, including types of potentially infinite values (streams etc).

Our full abstraction result cannot be extended to inequational full abstraction with respect to the natural order relation on the elements of probabilistic coherence spaces: a natural research direction will be to investigate other (pre)order relations and their possible interactive definitions. Also, it is quite tempting to replace the equality of probabilities in the definition of contextual equivalence by a distance; this clearly requires further developments.

## 2. PROBABILISTIC CALL BY PUSH VALUE

We introduce a probabilistic extension  $\Lambda_{\text{HP}}^{\text{P}}$  of CBPV (where HP stands for “half polarized”).

Types are given by the following BNF syntax. We define by mutual induction two kinds of types: *positive types* and *general types*, given type variables  $\zeta, \xi, \dots$ :

$$\text{positive } \varphi, \psi, \dots := \mathbf{1} \mid !\sigma \mid \varphi \otimes \psi \mid \varphi \oplus \psi \mid \zeta \mid \mathbf{Fix} \zeta \cdot \varphi \quad (2.1)$$

$$\text{general } \sigma, \tau, \dots := \varphi \mid \varphi \multimap \sigma \quad (2.2)$$

We consider the types up to the equation  $\mathbf{Fix} \zeta \cdot \varphi = \varphi [(\mathbf{Fix} \zeta \cdot \varphi) / \zeta]$ .

Terms are given by the following BNF syntax, given variables  $x, y, \dots$ :

$$\begin{aligned} M, N, \dots := & x \mid () \mid M^! \mid (M, N) \mid \text{in}_1 M \mid \text{in}_2 M \\ & \mid \lambda x^\varphi M \mid \langle M \rangle N \mid \text{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) \\ & \mid \text{pr}_1 M \mid \text{pr}_2 M \mid \text{der}(M) \mid \text{fix } x^{! \sigma} M \\ & \mid \text{fold}(M) \mid \text{unfold}(M) \\ & \mid \text{coin}(p), \ p \in [0, 1] \cap \mathbb{Q} \end{aligned}$$

This calculus can be seen as a version of Levy’s CBPV [15] in which the type constructor  $F$  is kept implicit (and  $U$  is “!”). It is close to SFPL [17]. We use LL inspired notations:  $M^!$  corresponds to  $\text{thunk}(M)$  and  $\text{der}(M)$  to  $\text{force}(M)$ .

Figure 1 provides the typing rules for these terms. A typing context is an expression  $\mathcal{P} = (x_1 : \varphi_1, \dots, x_k : \varphi_k)$  where all types are positive and the  $x_i$ s are pairwise distinct variables.

**2.1. Reduction rules.** *Values* are particular  $\Lambda_{\text{HP}}$  terms (they are not a new syntactic category) defined by the following BNF syntax:

$$V, W, \dots := x \mid () \mid M^! \mid (V, W) \mid \text{in}_1 V \mid \text{in}_2 V \mid \text{fold}(V).$$

Figure 2 defines a deterministic *weak* reduction relation  $\rightarrow_w$  and a probabilistic reduction  $\xrightarrow{p}$  relation. This reduction is weak in the sense that we never reduce within a “box”  $M^!$  or under a  $\lambda$ .

The distinguishing feature of this reduction system is the role played by values in the definition of  $\rightarrow_w$ . Consider for instance the case of the term  $\text{pr}_1(M_1, M_2)$ ; one might expect this term to reduce directly to  $M_1$  but this is not the case. One needs first to reduce  $M_1$  and  $M_2$  to values before extracting the first component of the pair (the terms  $\text{pr}_1(M_1, M_2)$  and  $M_1$  have not the same denotational interpretation in general). Of course replacing  $M_i$

$$\begin{array}{c}
\frac{}{\mathcal{P}, x : \varphi \vdash x : \varphi} \quad \frac{\mathcal{P}, x : \varphi \vdash M : \sigma}{\mathcal{P} \vdash \lambda x^\varphi M : \varphi \multimap \sigma} \quad \frac{\mathcal{P} \vdash M : \varphi \multimap \sigma \quad \mathcal{P} \vdash N : \varphi}{\mathcal{P} \vdash \langle M \rangle N : \sigma} \\
\frac{\mathcal{P} \vdash M : \sigma}{\mathcal{P} \vdash M^! : !\sigma} \quad \frac{}{\mathcal{P} \vdash () : \mathbf{1}} \quad \frac{\mathcal{P} \vdash M_1 : \varphi_1 \quad \mathcal{P} \vdash M_2 : \varphi_2}{\mathcal{P} \vdash (M_1, M_2) : \varphi_1 \otimes \varphi_2} \quad \frac{\mathcal{P} \vdash M : \varphi_i}{\mathcal{P} \vdash \text{in}_i M : \varphi_1 \oplus \varphi_2} \\
\frac{\mathcal{P} \vdash M : !\sigma}{\mathcal{P} \vdash \text{der}(M) : \sigma} \quad \frac{\mathcal{P} \vdash M : \varphi_1 \otimes \varphi_2}{\mathcal{P} \vdash \text{pr}_i M : \varphi_i} \quad \frac{\mathcal{P}, x : !\sigma \vdash M : \sigma}{\mathcal{P} \vdash \text{fix } x^{!\sigma} M : \sigma} \\
\frac{\mathcal{P} \vdash M : \varphi_1 \oplus \varphi_2 \quad \mathcal{P}, x_1 : \varphi_1 \vdash M_1 : \sigma \quad \mathcal{P}, x_2 : \varphi_2 \vdash M_2 : \sigma}{\mathcal{P} \vdash \text{case}(M, x_1 \cdot M_1, x_2 \cdot M_2) : \sigma} \\
\frac{}{\mathcal{P} \vdash \text{coin}(p) : \mathbf{1} \oplus \mathbf{1}} \\
\frac{\mathcal{P} \vdash M : \psi [\text{Fix } \zeta \cdot \psi / \zeta]}{\mathcal{P} \vdash \text{fold}(M) : \text{Fix } \zeta \cdot \psi} \quad \frac{\mathcal{P} \vdash M : \text{Fix } \zeta \cdot \psi}{\mathcal{P} \vdash \text{unfold}(M) : \psi [\text{Fix } \zeta \cdot \psi / \zeta]}
\end{array}$$

Figure 1: Typing system for  $\Lambda_{\text{HP}}$ 

with  $M_i^!$  allows a lazy behavior. Similarly, in the  $\rightarrow_w$  rule for **case**, the term on which the test is made must be reduced to a value (necessarily of shape  $\text{in}_i V$  if the expression is well typed) before the reduction is performed. As explained in the Introduction this allows to “memoize” the value  $V$  for further usage: the value is passed to the relevant branch of the **case** through the variable  $x_i$ .

We say that  $M$  is *weak normal* if there is no reduction  $M \xrightarrow{p} M'$ . It is clear that any value is weak normal. When  $M$  is closed,  $M$  is weak normal iff it is a value or an abstraction.

Given two terms  $M, M'$  and a real number  $p \in [0, 1]$ ,  $M \xrightarrow{p} M'$  means that  $M$  reduces in one step to  $M'$  with probability  $p$ .

In order to simplify the presentation we *choose* in Figure 2 a reduction strategy. For instance we decide that, for reducing  $(M_1, M_2)$  to a value, one needs first to reduce  $M_1$  to a value, and then  $M_2$ ; this choice is of course completely arbitrary. A similar choice is made for reducing terms of shape  $\langle M \rangle N$ , where we require the argument to be reduced first. This choice is less arbitrary as it will simplify a little bit the proof of adequacy in Section 3.4 (see for instance the proof of Lemma 21).

We could perfectly define a more general weak reduction relation as in [4] for which we could prove a “diamond” confluence property but we would then need to deal with a reduction transition system where, at each node (term), several probability distributions of reduced terms are available and so we would not be able to describe reduction as a simple (infinite dimensional) stochastic matrix. We could certainly also define more general reduction rules allowing to reduce redexes anywhere in terms (apart for  $\text{coin}(p)$  which can be reduced only when in linear position) but this would require the introduction of additional  $\sigma$ -rules as in [5]. As in that paper, confluence can probably be proven, using ideas coming from [7, 21] for dealing with reduction in an algebraic lambda-calculus setting.

**2.2. Observational equivalence.** In order to define observational equivalence, we need to represent the probability of convergence of a term to a normal form. As in [2], we consider the reduction as a discrete time Markov chain whose states are terms and stationary states are weak normal terms. We then define a stochastic matrix  $\text{Red} \in [0, 1]^{\Lambda_{\text{HP}}^p \times \Lambda_{\text{HP}}^p}$  (indexed by

$$\begin{array}{c}
\frac{}{\text{der}(M^!) \rightarrow_w M} \quad \frac{}{\langle \lambda x^\varphi M \rangle V \rightarrow_w M[V/x]} \quad \frac{}{\text{pr}_i(V_1, V_2) \rightarrow_w V_i} \\
\frac{}{\text{fix } x^{! \sigma} M \rightarrow_w M[(\text{fix } x^{! \sigma} M)^! / x]} \quad \frac{}{\text{case}(\text{in}_i V, x_1 \cdot M_1, x_2 \cdot M_2) \rightarrow_w M_i[V/x_i]} \\
\frac{}{\text{unfold}(\text{fold}(V)) \rightarrow_w V} \\
\frac{M \rightarrow_w M'}{M \xrightarrow{1} M'} \quad \frac{}{\text{coin}(p) \xrightarrow{p} \text{in}_1()} \quad \frac{}{\text{coin}(p) \xrightarrow{1-p} \text{in}_2()} \\
\frac{M \xrightarrow{p} M'}{\text{der}(M) \xrightarrow{p} \text{der}(M')} \quad \frac{M \xrightarrow{p} M'}{\langle M \rangle V \xrightarrow{p} \langle M' \rangle V} \quad \frac{N \xrightarrow{p} N'}{\langle M \rangle N \xrightarrow{p} \langle M \rangle N'} \quad \frac{M \xrightarrow{p} M'}{\text{pr}_i M \xrightarrow{p} \text{pr}_i M'} \\
\frac{M_1 \xrightarrow{p} M'_1}{(M_1, M_2) \xrightarrow{p} (M'_1, M_2)} \quad \frac{M_2 \xrightarrow{p} M'_2}{(V, M_2) \xrightarrow{p} (V, M'_2)} \quad \frac{M \xrightarrow{p} M'}{\text{in}_i M \xrightarrow{p} \text{in}_i M'} \\
\frac{M \xrightarrow{p} M'}{\text{case}(M, x_1 \cdot M_1, x_2 \cdot M_2) \xrightarrow{p} \text{case}(M', x_1 \cdot M_1, x_2 \cdot M_2)} \\
\frac{M \xrightarrow{p} M'}{\text{fold}(M) \xrightarrow{p} \text{fold}(M')} \quad \frac{M \xrightarrow{p} M'}{\text{unfold}(M) \xrightarrow{p} \text{unfold}(M')}
\end{array}$$

Figure 2: Weak and Probabilistic reduction axioms and rules for  $\Lambda_{\text{HP}}^p$ 

terms) as

$$\text{Red}_{M, M'} = \begin{cases} p & \text{if } M \xrightarrow{p} M' \\ 1 & \text{if } M \text{ is weak-normal and } M' = M \\ 0 & \text{otherwise.} \end{cases}$$

Saying that  $\text{Red}$  is stochastic means that the coefficients of  $\text{Red}$  belong to  $[0, 1]$  and that, for any given term  $M$ , one has  $\sum_{M'} \text{Red}_{M, M'} = 1$  (actually there are at most two terms  $M'$  such that  $\text{Red}_{M, M'} \neq 0$ ).

Then, for all  $M, M' \in \Lambda_{\text{HP}}^p$ , if  $M'$  is weak-normal then the sequence  $(\text{Red}_{M, M'}^n)_{n=1}^\infty$  is monotone and included in  $[0, 1]$ , and therefore has a lub that we denote as  $\text{Red}_{M, M'}^\infty$  which defines a sub-stochastic matrix (taking  $\text{Red}_{M, M'}^\infty = 0$  when  $M'$  is not weak-normal).

When  $M'$  is weak-normal, the number  $p = \text{Red}_{M, M'}^\infty$  is the probability that  $M$  reduces to  $M'$  after a finite number of steps.

Let us say when two closed terms  $M_1, M_2$  of type  $\sigma$  are *observationally equivalent*:

$$M_1 \sim M_2, \text{ if for all closed term } C \text{ of type } !\sigma \multimap \mathbf{1}, \text{Red}_{\langle C \rangle M_1^!, ()}^\infty = \text{Red}_{\langle C \rangle M_2^!, ()}^\infty.$$

For simplicity we consider only closed terms  $M_1$  and  $M_2$ . We could also define an observational equivalence on non closed terms, replacing the term  $C$  with a context  $C[\ ]$  which could bind free variables of the  $M_i$ 's, this would not change the results of the paper.

### 2.3. Examples.

*Ever-looping program.* Given any type  $\sigma$ , we define  $\Omega^\sigma = \text{fix } x^{! \sigma} \text{der}(x)$  which satisfies  $\vdash \Omega^\sigma : \sigma$ . It is clear that  $\Omega^\sigma \rightarrow_w \text{der}((\Omega^\sigma)^!) \rightarrow_w \Omega^\sigma$  so that we can consider  $\Omega^\sigma$  as the ever-looping program of type  $\sigma$ .

*Booleans.* We define the type  $o = \mathbf{1} \oplus \mathbf{1}$ . We define the “true” constant as  $\mathbf{t} = \text{in}_1()$  and the “false” constant as  $\mathbf{f} = \text{in}_2()$ .

*Natural numbers.* We define the type  $\iota$  of unary natural numbers by  $\iota = \mathbf{1} \oplus \iota$  (by this we mean that  $\iota = \mathbf{Fix} \, \zeta \cdot (\mathbf{1} \oplus \zeta)$ ). We define  $\underline{0} = \text{in}_1()$  and  $\underline{n+1} = \text{in}_2 \underline{n}$  so that we have  $\mathcal{P} \vdash \underline{n} : \iota$  for each  $n \in \mathbb{N}$ .

Then, given a term  $M$ , we define the term  $\text{suc}(M) = \text{in}_2 M$ , so that we have

$$\frac{\mathcal{P} \vdash M : \iota}{\mathcal{P} \vdash \text{suc}(M) : \iota}$$

Last, given terms  $M$ ,  $N_1$  and  $N_2$  and a variable  $x$ , we define an “ifz” conditional by  $\text{if}(M, N_1, (x)N_2) = \text{case}(M, z \cdot N_1, x \cdot N_2)$  where  $z$  is not free in  $N_1$ , so that

$$\frac{\mathcal{P} \vdash M : \iota \quad \mathcal{P} \vdash N_1 : \sigma \quad \mathcal{P}, x : \iota \vdash N_2 : \sigma}{\mathcal{P} \vdash \text{if}(M, N_1, (x)N_2) : \sigma}$$

We exhibit terms using this constructions in the paragraph on probabilistic tests just below. The first ones  $\text{dice}_p(M_1, M_2)$  and  $\text{ran}(\vec{p})$  do not use the possibility to save the value of the test. The second one  $\text{prob}_k$  exploits this call-by-value feature.

*Streams.* Let  $\varphi$  be a positive type and  $S_\varphi$  be the positive type defined by  $S_\varphi = \varphi \otimes !S_\varphi$ , that is  $S_\varphi = \mathbf{Fix} \, \zeta \cdot (\varphi \otimes !\zeta)$ . We can define a term  $M$  such that  $\vdash M : S_\varphi \multimap \iota \multimap \varphi$  which computes the  $n$ th element of a stream:

$$M = \text{fix } f^{!(S_\varphi \multimap \iota \multimap \varphi)} \lambda x^{S_\varphi} \lambda y^\iota \text{if}(y, \text{pr}_1 x, (z) \langle \text{der}(f) \rangle \text{der}(\text{pr}_2 x) z)$$

Conversely, we can define a term  $N$  such that  $\vdash N : !(\iota \multimap \varphi) \multimap S_\varphi$  which turns a function into the stream of its successive application to an integer.

$$N = \text{fix } F^{!(\iota \multimap \varphi) \multimap S_\varphi} \lambda f^{!(\iota \multimap \varphi)} \left( \langle \text{der}(f) \rangle \underline{0}, (\langle \text{der}(F) \rangle (\lambda x^\iota \langle \text{der}(f) \rangle \text{suc}(x))^\iota)^\iota \right)$$

Observe that the recursive call of  $F$  is encapsulated into a box, which makes the construction lazy.

*Lists.* There are various possibilities for defining a type of lists of elements of a positive type  $\varphi$ . The simplest definition is  $\lambda_0 = \mathbf{1} \oplus (\varphi \otimes \lambda_0)$ . This corresponds to the ordinary ML type of lists. But we can also define  $\lambda_1 = \mathbf{1} \oplus (\varphi \otimes !\lambda_1)$  and then we have a type of lazy lists (or terminable streams) where the tail of the list is computed only when required.

We could also consider  $\lambda_2 = \mathbf{1} \oplus (!\sigma \otimes \lambda_2)$  which allows to manipulate lists of objects of type  $\sigma$  (which can be a general type) without accessing their elements.

*Probabilistic tests.* If  $\mathcal{P} \vdash M_i : \sigma$  for  $i = 1, 2$ , we set  $\text{dice}_p(M_1, M_2) = \text{if}(\text{coin}(p), M_1, (z)M_2)$  (where  $z$  is not free in  $M_2$ ) and this term satisfies  $\mathcal{P} \vdash \text{dice}_p(M_1, M_2) : \sigma$ . If  $M_i$  reduces to a value  $V_i$  with probability  $q_i$ , then  $\text{dice}_p(M_1, M_2)$  reduces to  $V_i$  with probability  $p q_i$ .

Let  $n \in \mathbb{N}$  and let  $\vec{p} = (p_0, \dots, p_n)$  be such that  $p_i \in [0, 1] \cap \mathbb{Q}$  and  $p_0 + \dots + p_n \leq 1$ . Then one defines a closed term  $\text{ran}(\vec{p})$ , such that  $\vdash \text{ran}(\vec{p}) : \iota$ , which reduces to  $\underline{i}$  with probability  $p_i$  for each  $i \in \{0, \dots, n\}$ . The definition is by induction on  $n$ .

$$\text{ran}(\vec{p}) = \begin{cases} \underline{0} & \text{if } p_0 = 1 \text{ whatever be the value of } n \\ \text{if}(\text{coin}(p_0), \underline{0}, (z)\Omega^\iota) & \text{if } n = 0 \\ \text{if}(\text{coin}(p_0), \underline{0}, (z) \text{succ}(\text{ran}(\frac{p_1}{1-p_0}, \dots, \frac{p_n}{1-p_0}))) & \text{otherwise} \end{cases}$$



As another example of use of the conditional, we define, by induction on  $k$ , a family of terms  $\text{prob}_k$  such that  $\vdash \text{prob}_k : \iota \multimap \mathbf{1}$ :

$$\text{prob}_0 = \lambda x^\iota \text{if}(x, (), (z)\Omega^\iota) \quad \text{prob}_{k+1} = \lambda x^\iota \text{if}(x, \Omega^\iota, (z)\langle \text{prob}_k \rangle z)$$

For  $M$  such that  $\vdash M : \iota$ , the term  $\langle \text{prob}_k \rangle M$  reduces to  $()$  with a probability which is equal to the probability of  $M$  to reduce to  $\underline{k}$ .

Now, we introduce terms that will be used in the definition of testing terms the proof of Full Abstraction in Section 4.

First, we define  $\text{prod}_k$  such that  $\vdash \text{prod}_k : 1^{\otimes k} \multimap \varphi \multimap \varphi$ :

$$\text{prod}_0 = \lambda y^\varphi y \quad \text{prod}_{k+1} = \lambda x^{\mathbf{1}} \text{prod}_k.$$

Given  $M_1, \dots, M_k$  such that  $\mathcal{P} \vdash M_i : \mathbf{1}$  and  $\mathcal{P} \vdash N : \varphi$ , the term  $\langle \langle \text{prod}_{k+1} \rangle M_1 \dots \rangle M_k \rangle N$  reduces to a value  $V$  with probability  $p_1 \dots p_k q$  where  $p_i$  is the probability of  $M_i$  to reduce to  $()$  and  $q$  is the probability of  $N$  to reduce to  $V$ . We use the notations:

$$M_1 \cdot N = \langle \langle \text{prod}_1 \rangle M_1 \rangle N \quad M_1 \wedge \dots \wedge M_k = \langle \langle \text{prod}_k \rangle M_1 \dots \rangle M_k,$$

so that  $\mathcal{P} \vdash M_1 \cdot N : \varphi$  and the probability that  $M_1 \cdot N$  reduces to  $V$  is  $p_1 q$  and  $\mathcal{P} \vdash M_1 \wedge \dots \wedge M_k : \mathbf{1}$  and  $M_1 \wedge \dots \wedge M_k$  reduces to  $()$  with probability  $p_1 \dots p_k$ .

*Notation.* Given terms  $M_1, \dots, M_n$ , we define a term  $N = (M_1, \dots, M_n)$  by induction on  $n$ : if  $n = 0$  then  $N = ()$  and if  $n > 0$  then  $N = ((M_1, \dots, M_{n-1}), M_n)$ . We freely identify  $(M)$  with  $M$  and extend the notation  $\text{pr}_i N$  in the obvious way.

Given a positive type  $\varphi$  and  $k \in \mathbb{N}$ , we define a term  $\text{choose}_k$  such that  $\vdash \text{choose}_k : \iota \multimap \varphi^{\otimes k} \multimap \varphi$

$$\text{choose}_0 = \lambda \xi^\iota \Omega^\varphi$$

$$\text{choose}_{k+1} = \lambda \xi^\iota \lambda x_0^\varphi \dots \lambda x_k^\varphi \text{if}(\xi, x_0, (z)\langle \langle \text{choose}_k \rangle z \rangle (x_1, \dots, x_k)).$$

Given a  $P$  such that  $\mathcal{P} \vdash P : \iota$  and terms  $N_0, \dots, N_k$  such that  $\mathcal{P} \vdash N_i : \varphi$  for any  $i$ , we use the notation,

$$\overline{\sum_{i=0}^k} P_{\underline{i}} \cdot N_i = \langle \langle \text{choose}_{k+1} \rangle P \rangle (N_0, \dots, N_k).$$

The term  $\overline{\sum_{i=0}^k} P_{\underline{i}} \cdot N_i$  reduces to  $V_i$  with probability  $p_i q_i$  where  $p_i$  is the probability that  $P$  reduces to  $\underline{i}$  which is symbolized by the notation  $P_{\underline{i}}$  and  $q_i$  is the probability that  $N_i$  reduces to  $V_i$ .

As we will see more precisely in Paragraph 3.2.12, a term of type  $\iota$  can be seen as a sub-probability distribution over  $\mathbb{N}$ . Given  $1 \leq l \leq r$ , we introduce the term of type  $\iota \multimap \iota$ :

$$\text{ext}(l, r) = \lambda z^\iota \langle \langle \text{choose}_r \rangle z \rangle (\underbrace{\Omega^\iota, \dots, \Omega^\iota}_{l-1}, \underline{l}, \dots, \underline{r}).$$

such that if  $\vdash P : \iota$ , then  $\langle \text{ext}(l, r) \rangle P$  extracts the sub-probability distribution with support between  $l$  and  $r$ . Indeed,  $\langle \text{ext}(l, r) \rangle P$  reduces to  $\overline{\sum_{i=l}^r} P_{\underline{i}} \cdot \underline{i}$ .

We also introduce, for  $\vec{n} = (n_1, \dots, n_k)$  a sequence of  $k$  natural numbers, a term  $\text{win}_i(\vec{n})$  of type  $\iota \multimap \iota$  which extracts the sub-probability distribution whose support is in the  $i^{\text{th}}$  window of length  $n_i$ :

$$\text{win}_i(\vec{n}) = \text{ext}(n_1 + \dots + n_{i-1} + 1, n_1 + \dots + n_i).$$

### 3. PROBABILISTIC COHERENT SPACES

**3.1. Semantics of LL, in a nutshell.** The kind of denotational models we are interested in, in this paper, are those induced by a model of LL, as explained in [4]. We remind the basic definitions and notations, referring to that paper for more details.

**3.1.1. Models of Linear Logic.** A model of LL consists of the following data.

A symmetric monoidal closed category  $(\mathcal{L}, \otimes, 1, \lambda, \rho, \alpha, \sigma)$  where we use simple juxtaposition  $gf$  to denote composition of morphisms  $f \in \mathcal{L}(X, Y)$  and  $g \in \mathcal{L}(Y, Z)$ . We use  $X \multimap Y$  for the object of linear morphisms from  $X$  to  $Y$ ,  $\text{ev} \in \mathcal{L}((X \multimap Y) \otimes X, Y)$  for the evaluation morphism and  $\text{cur} \in \mathcal{L}(Z \otimes X, Y) \rightarrow \mathcal{L}(Z, X \multimap Y)$  for the linear curryfication map. For convenience, and because it is the case in the concrete models we consider (such as Scott Semantics [4] or Probabilistic Coherent Spaces here), we assume this SMCC to be a  $*$ -autonomous category with dualizing object  $\perp$ . We use  $X^\perp$  for the object  $X \multimap \perp$  of  $\mathcal{L}$  (the dual, or linear negation, of  $X$ ).

$\mathcal{L}$  is cartesian with terminal object  $\top$ , product  $\&$ , projections  $\text{pr}_i$ . By  $*$ -autonomy  $\mathcal{L}$  is co-cartesian with initial object  $0$ , coproduct  $\oplus$  and injections  $\text{in}_i$ .

We are given a comonad  $!_- : \mathcal{L} \rightarrow \mathcal{L}$  with co-unit  $\text{der}_X \in \mathcal{L}(!X, X)$  (*dereliction*) and co-multiplication  $\text{dig}_X \in \mathcal{L}(!X, !!X)$  (*digging*) together with a strong symmetric monoidal structure (Seely isos  $\text{m}^0$  and  $\text{m}^2$ ) for the functor  $!_-$ , from the symmetric monoidal category  $(\mathcal{L}, \&)$  to the symmetric monoidal category  $(\mathcal{L}, \otimes)$  satisfying an additional coherence condition wrt.  $\text{dig}$ .

We use  $?_-$  for the “De Morgan dual” of  $!_-$ :  $?X = (!X^\perp)^\perp$  and similarly for morphisms. It is a monad on  $\mathcal{L}$ .

**3.1.2. The Eilenberg-Moore category.** It is then standard to define the category  $\mathcal{L}^!$  of  $!_-$ -coalgebras. An object of this category is a pair  $P = (\underline{P}, h_P)$  where  $\underline{P} \in \text{Obj}(\mathcal{L})$  and  $h_P \in \mathcal{L}(\underline{P}, !\underline{P})$  is such that  $\text{der}_{\underline{P}} h_P = \text{Id}$  and  $\text{dig}_{\underline{P}} h_P = !h_P h_P$ . Then  $f \in \mathcal{L}^!(P, Q)$  iff  $f \in \mathcal{L}(\underline{P}, \underline{Q})$  such that  $h_Q f = !f h_P$ . The functor  $!_-$  can be seen as a functor from  $\mathcal{L}$  to  $\mathcal{L}^!$  mapping  $\underline{X}$  to  $(!X, \text{dig}_X)$  and  $f \in \mathcal{L}(X, Y)$  to  $!f$ . It is right adjoint to the forgetful functor  $U : \mathcal{L}^! \rightarrow \mathcal{L}$ . Given  $f \in \mathcal{L}(\underline{P}, X)$ , we use  $f^! \in \mathcal{L}^!(P, !X)$  for the morphism associated with  $f$  by this adjunction, one has  $f^! = !f h_P$ . If  $g \in \mathcal{L}^!(Q, P)$ , we have  $f^! g = (f g)^!$ .

Then  $\mathcal{L}^!$  is cartesian (with product of shape  $P \otimes Q = (\underline{P} \otimes \underline{Q}, h_{P \otimes Q})$  and terminal object  $(1, h_1)$ , still denoted as  $1$ ). This category is also co-cartesian with coproduct of shape  $P \oplus Q = (\underline{P} \oplus \underline{Q}, h_{P \oplus Q})$  and initial object  $(0, h_0)$  still denoted as  $0$ . The complete definitions can be found in [4]. We use  $c_P \in \mathcal{L}^!(P, P \otimes P)$  (*contraction*) for the diagonal and  $w_P \in \mathcal{L}^!(P, 1)$  (*weakening*) for the unique morphism to the terminal object.

We also consider occasionally the *Kleisli category*<sup>3</sup>  $\mathcal{L}_!$  of the comonad  $!$ : its objects are those of  $\mathcal{L}$  and  $\mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$ . The identity at  $X$  in this category is  $\text{der}_X$  and composition of  $f \in \mathcal{L}_!(X, Y)$  and  $g \in \mathcal{L}_!(Y, Z)$  is defined as

$$g \circ f = g!f \text{ dig}_X.$$

This category is cartesian closed but this fact will not play an essential role in this work.

**3.1.3. Fix-points.** For any object  $X$ , we assume to be given  $\text{fix}_X \in \mathcal{L}(!(X \multimap X), X)$ , a morphism such that<sup>4</sup>  $\text{ev}(\text{der}_{!X \multimap X} \otimes \text{fix}_X) \circ \text{c}_{!(X \multimap X)} = \text{fix}_X$  which will allow to interpret term fix-points.

In order to interpret fix-points of types, we assume that the category  $\mathcal{L}$  is equipped with a notion of embedding-retraction pairs, following a standard approach. We use  $\mathcal{L}_\subseteq$  for the corresponding category. It is equipped with a functor  $F : \mathcal{L}_\subseteq \rightarrow \mathcal{L}^{\text{op}} \times \mathcal{L}$  such that  $F(X) = (X, X)$  and for which we use the notation  $(\varphi^-, \varphi^+) = F(\varphi)$  and assume that  $\varphi^- \varphi^+ = \text{Id}_X$ . We assume furthermore that  $\mathcal{L}_\subseteq$  has all countable directed colimits and that the functor  $E = \text{pr}_2 F : \mathcal{L}_\subseteq \rightarrow \mathcal{L}$  is continuous. We also assume that all the basic operations on objects  $(\otimes, \oplus, (-)^\perp$  and  $!$ ) are continuous functors from  $\mathcal{L}_\subseteq$  to itself<sup>5</sup>.

Then it is easy to carry this notion of embedding-retraction pairs to  $\mathcal{L}^!$ , defining a category  $\mathcal{L}_\subseteq^!$ , to show that this category has all countable directed colimits and that the functors  $\otimes$  and  $\oplus$  are continuous on this category:  $\mathcal{L}_\subseteq^!(P, Q)$  is the set of all  $\varphi \in \mathcal{L}_\subseteq(P, Q)$  such that  $\varphi^+ \in \mathcal{L}^!(P, Q)$ . One checks also that  $!$  defines a continuous functor from  $\mathcal{L}_\subseteq$  to  $\mathcal{L}_\subseteq^!$ . This allows to interpret recursive types, more details can be found in [3].

**3.1.4. Interpreting types.** Using straightforwardly the object  $1$  and the operations  $\otimes, \oplus, !$  and  $\multimap$  of the model  $\mathcal{L}$  as well as the completeness and continuity properties explained in Section 3.1.3, we associate with any positive type  $\varphi$  and any repetition-free list  $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$  of type variables containing all free variables of  $\varphi$  a continuous functor  $[\varphi]_{\vec{\zeta}}^! : (\mathcal{L}_\subseteq^!)^n \rightarrow \mathcal{L}_\subseteq^!$  and with any general type  $\sigma$  and any list  $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$  of pairwise distinct type variables containing all free variables of  $\sigma$  we associate a continuous functor  $[\sigma]_{\vec{\zeta}} : (\mathcal{L}_\subseteq^!)^n \rightarrow \mathcal{L}_\subseteq$ .

When we write  $[\sigma]$  or  $[\varphi]^!$  (without subscript), we assume implicitly that the types  $\sigma$  and  $\varphi$  have no free type variables. Then  $[\sigma]$  is an object of  $\mathcal{L}$  and  $[\varphi]^!$  is an object of  $\mathcal{L}^!$ . We have  $[\varphi] = [\varphi]^!$  that is, considered as a generalized type, the semantics of a positive type  $\varphi$  is the carrier of the coalgebra  $[\varphi]^!$ .

Given a typing context  $\mathcal{P} = (x_1 : \varphi_1, \dots, x_k : \varphi_k)$ , we define  $[\mathcal{P}] = [\varphi_1]^! \otimes \dots \otimes [\varphi_k]^! \in \mathcal{L}^!$ .

In the model or probabilistic coherence spaces considered in this paper, we define  $\mathcal{L}_\subseteq$  in such a way that the only isos are the identity maps. This implies that the types  $\text{Fix } \zeta \cdot \varphi$  and  $\varphi[(\text{Fix } \zeta \cdot \varphi)/\zeta]$  are interpreted as *the same object* (or functor). Such definitions of  $\mathcal{L}_\subseteq$  are possible in many other models (relational, coherence spaces, hypercoherences etc).

<sup>3</sup>It is the full subcategory of  $\mathcal{L}^!$  of free coalgebras, see any introductory text on monads and co-monads.

<sup>4</sup>It might seem natural to require the stronger uniformity conditions of *Conway operator* [20]. This does not seem to be necessary as far as soundness of our semantics is concerned even if the fix-point operators arising in concrete models satisfy these further properties.

<sup>5</sup>This is a rough statement; one has to say for instance that if  $\varphi_i \in \mathcal{L}_\subseteq(X_i, Y_i)$  for  $i = 1, 2$  then  $(\varphi_1 \otimes \varphi_2)^- = \varphi_1^- \otimes \varphi_2^-$  etc. The details can be found in [4].

We postpone the description of term interpretation because this will require constructions specific to our probabilistic semantics, in addition to the generic categorical ingredients introduced so far.

**3.2. The model of probabilistic coherence spaces.** Given a countable set  $I$  and  $u, u' \in (\mathbb{R}^+)^I$ , we set  $\langle u, u' \rangle = \sum_{i \in I} u_i u'_i$ . Given  $\mathcal{F} \subseteq (\mathbb{R}^+)^I$ , we set  $\mathcal{F}^\perp = \{u' \in (\mathbb{R}^+)^I \mid \forall u \in \mathcal{F} \langle u, u' \rangle \leq 1\}$ .

A *probabilistic coherence space* (PCS) is a pair  $X = (|X|, \mathbf{P}X)$  where  $|X|$  is a countable set and  $\mathbf{P}X \subseteq (\mathbb{R}^+)^{|X|}$  satisfies

- $\mathbf{P}X^{\perp\perp} = \mathbf{P}X$  (equivalently,  $\mathbf{P}X^{\perp\perp} \subseteq \mathbf{P}X$ ),
- for each  $a \in |X|$  there exists  $u \in \mathbf{P}X$  such that  $u_a > 0$ ,
- for each  $a \in |X|$  there exists  $A > 0$  such that  $\forall u \in \mathbf{P}X \ u_a \leq A$ .

If only the first of these conditions holds, we say that  $X$  is a *pre-probabilistic coherence space* (pre-PCS).

The purpose of the second and third conditions is to prevent infinite coefficients to appear in the semantics. This property in turn will be essential for guaranteeing the morphisms interpreting proofs to be analytic functions, which will be the key property to prove full abstraction. So these conditions, though cosmetics at first sight, are important for our ultimate goal.

**Lemma 1.** *Let  $X$  be a pre-PCS. The following conditions are equivalent:*

- $X$  is a PCS,
- $\forall a \in |X| \exists u \in \mathbf{P}X \exists u' \in \mathbf{P}X^\perp \ u_a > 0 \text{ and } u'_a > 0$ ,
- $\forall a \in |X| \exists A > 0 \forall u \in \mathbf{P}X \forall u' \in \mathbf{P}X^\perp \ u_a \leq A \text{ and } u'_a \leq A$ .

The proof is straightforward.

We equip  $\mathbf{P}X$  with the most obvious partial order relation:  $u \leq v$  if  $\forall a \in |X| \ u_a \leq v_a$  (using the usual order relation on  $\mathbb{R}$ ).

**Theorem 2.**  *$\mathbf{P}X$  is an  $\omega$ -continuous domain. Given  $u, v \in \mathbf{P}X$  and  $\alpha, \beta \in \mathbb{R}^+$  such that  $\alpha + \beta \leq 1$ , one has  $\alpha u + \beta v \in \mathbf{P}X$ .*

This is an easy consequence of the hypothesis  $\mathbf{P}X^{\perp\perp} \subseteq \mathbf{P}X$ . See [2] for details.

**3.2.1. Morphisms of PCSs.** Let  $X$  and  $Y$  be PCSs. Let  $t \in (\mathbb{R}^+)^{|X| \times |Y|}$  (to be understood as a matrix). Given  $u \in \mathbf{P}X$ , we define  $tu \in \overline{\mathbb{R}^+}^{|Y|}$  by  $(tu)_b = \sum_{a \in |X|} t_{a,b} u_a$  (application of the matrix  $t$  to the vector  $u$ )<sup>6</sup>. We say that  $t$  is a *(linear) morphism* from  $X$  to  $Y$  if  $\forall u \in \mathbf{P}X \ tu \in \mathbf{P}Y$ , that is

$$\forall u \in \mathbf{P}X \forall v' \in \mathbf{P}Y^\perp \quad \sum_{(a,b) \in |X| \times |Y|} t_{a,b} u_a v'_b \leq 1.$$

The diagonal matrix  $\text{Id} \in (\mathbb{R}^+)^{|X| \times |X|}$  given by  $\text{Id}_{a,b} = 1$  if  $a = b$  and  $\text{Id}_{a,b} = 0$  otherwise is a morphism. In that way we have defined a category **Pcoh** whose objects are the PCSs and

<sup>6</sup>This is an unordered sum, which is infinite in general. It makes sense because all its terms are  $\geq 0$ .

whose morphisms have just been defined. Composition of morphisms is defined as matrix multiplication: let  $s \in \mathbf{Pcoh}(X, Y)$  and  $t \in \mathbf{Pcoh}(Y, Z)$ , we define  $ts \in (\mathbb{R}^+)^{|X| \times |Z|}$  by

$$(ts)_{a,c} = \sum_{b \in |Y|} s_{a,b} t_{b,c}$$

and a simple computation shows that  $ts \in \mathbf{Pcoh}(X, Z)$ . More precisely, we use the fact that, given  $u \in PX$ , one has  $(ts)u = t(su)$ . Associativity of composition holds because matrix multiplication is associative.  $\text{Id}_X$  is the identity morphism at  $X$ .

Given  $u \in PX$ , we define  $\|u\|_X = \sup\{\langle u, u' \rangle \mid u' \in PX^\perp\}$ . By definition, we have  $\|u\|_X \in [0, 1]$ .

**3.2.2. Multiplicative constructs.** One sets  $X^\perp = (|X|, PX^\perp)$ . It results straightforwardly from the definition of PCSs that  $X^\perp$  is a PCS. Given  $t \in \mathbf{Pcoh}(X, Y)$ , one has  $t^\perp \in \mathbf{Pcoh}(Y^\perp, X^\perp)$  if  $t^\perp$  is the transpose of  $t$ , that is  $(t^\perp)_{b,a} = t_{a,b}$ .

One defines  $X \otimes Y$  by  $|X \otimes Y| = |X| \times |Y|$  and

$$P(X \otimes Y) = \{u \otimes v \mid u \in PX \text{ and } v \in PY\}^{\perp\perp}$$

where  $(u \otimes v)_{(a,b)} = u_a v_b$ . Then  $X \otimes Y$  is a pre-PCS.

We have

$$P(X \otimes Y^\perp)^\perp = \{u \otimes v' \mid u \in PX \text{ and } v' \in PY^\perp\}^\perp = \mathbf{Pcoh}(X, Y).$$

It follows that  $X \multimap Y = (X \otimes Y^\perp)^\perp$  is a pre-PCS. Let  $(a, b) \in |X| \times |Y|$ . Since  $X$  and  $Y^\perp$  are PCSs, there is  $A > 0$  such that  $u_a v'_b < A$  for all  $u \in PX$  and  $v' \in PY^\perp$ . Let  $t \in (\mathbb{R}^+)^{|X \multimap Y|}$  be such that  $t_{(a', b')} = 0$  for  $(a', b') \neq (a, b)$  and  $t_{(a, b)} = 1/A$ , we have  $t \in P(X \multimap Y)$ . This shows that  $\exists t \in P(X \multimap Y)$  such that  $t_{(a, b)} > 0$ . Similarly we can find  $u \in PX$  and  $v' \in PY^\perp$  such that  $\varepsilon = u_a v'_b > 0$ . It follows that  $\forall t \in P(X \multimap Y)$  one has  $t_{(a, b)} \leq 1/\varepsilon$ . We conclude that  $X \multimap Y$  is a PCS, and therefore  $X \otimes Y$  is also a PCS.

**Lemma 3.** *Let  $X$  and  $Y$  be PCSs. One has  $P(X \multimap Y) = \mathbf{Pcoh}(X, Y)$ . That is, given  $t \in (\mathbb{R}^+)^{|X| \times |Y|}$ , one has  $t \in P(X \multimap Y)$  iff for all  $u \in PX$ , one has  $tu \in PY$ .*

This results immediately from the definition above of  $X \multimap Y$ .

**Lemma 4.** *Let  $X_1, X_2$  and  $Y$  be PCSs. Let  $t \in (\mathbb{R}^+)^{|X_1 \otimes X_2 \multimap Y|}$ . One has  $t \in \mathbf{Pcoh}(X_1 \otimes X_2, Y)$  iff for all  $u_1 \in PX_1$  and  $u_2 \in PX_2$  one has  $t(u_1 \otimes u_2) \in PY$ .*

*Proof.* The condition stated by the lemma is clearly necessary. Let us prove that it is sufficient: under this condition, it suffices to prove that

$$t^\perp \in \mathbf{Pcoh}(Y^\perp, (X_1 \otimes X_2)^\perp).$$

Let  $v' \in PY^\perp$ , it suffices to prove that  $t^\perp v' \in P(X_1 \otimes X_2)^\perp$ . So let  $u_1 \in PX_1$  and  $u_2 \in PX_2$ , it suffices to prove that  $\langle t^\perp v', u_1 \otimes u_2 \rangle \leq 1$ , that is  $\langle t(u_1 \otimes u_2), v' \rangle \leq 1$ , which follows from our assumption.  $\square$

Let  $s_i \in \mathbf{Pcoh}(X_i, Y_i)$  for  $i = 1, 2$ . Then one defines

$$s_1 \otimes s_2 \in (\mathbb{R}^+)^{|X_1 \otimes X_2 \multimap Y_1 \otimes Y_2|}$$

by  $(s_1 \otimes s_2)_{((a_1, a_2), (b_1, b_2))} = (s_1)_{(a_1, b_1)} (s_2)_{(a_2, b_2)}$  and one must check that

$$s_1 \otimes s_2 \in \mathbf{Pcoh}(X_1 \otimes X_2, Y_1 \otimes Y_2).$$

This follows directly from Lemma 4. Let  $1 = (\{*\}, [0, 1])$ . There are obvious choices of natural isomorphisms

$$\begin{aligned}\lambda_X &\in \mathbf{Pcoh}(1 \otimes X, X) \\ \rho_X &\in \mathbf{Pcoh}(X \otimes 1, X) \\ \alpha_{X_1, X_2, X_3} &\in \mathbf{Pcoh}((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3)) \\ \sigma_{X_1, X_2} &\in \mathbf{Pcoh}(X_1 \otimes X_2, X_2 \otimes X_1)\end{aligned}$$

which satisfy the standard coherence properties. This shows that  $(\mathbf{Pcoh}, 1, \lambda, \rho, \alpha, \sigma)$  is a symmetric monoidal category.

**3.2.3. Internal linear hom.** Given PCSs  $X$  and  $Y$ , let us define  $\text{ev} \in (\mathbb{R}^+)^{|(X \multimap Y) \otimes X \multimap Y|}$  by

$$\text{ev}_{(((a', b'), a), b)} = \begin{cases} 1 & \text{if } (a, b) = (a', b') \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that  $(X \multimap Y, \text{ev})$  is an internal linear hom object in  $\mathbf{Pcoh}$ , showing that this SMCC is closed. If  $t \in \mathbf{Pcoh}(Z \otimes X, Y)$ , the corresponding linearly curried morphism  $\text{cur}(t) \in \mathbf{Pcoh}(Z, X \multimap Y)$  is given by  $\text{cur}(t)_{(c, (a, b))} = t_{((c, a), b)}$ .

**3.2.4. \*-autonomy.** Take  $\perp = 1$ , then one checks readily that  $(\mathbf{Pcoh}, 1, \lambda, \rho, \alpha, \sigma, \perp)$  is a \*-autonomous category. The duality functor  $X \mapsto (X \multimap \perp)$  can be identified with the strictly involutive contravariant functor  $X \mapsto X^\perp$ .

**3.2.5. Additives.** Let  $(X_i)_{i \in I}$  be a countable family of PCSs. We define a PCS  $\&_{i \in I} X_i$  by  $|\&_{i \in I} X_i| = \bigcup_{i \in I} \{i\} \times |X_i|$  and  $u \in \mathbf{P}(\&_{i \in I} X_i)$  if, for all  $i \in I$ , the family  $u(i) \in (\mathbb{R}^+)^{|X_i|}$  defined by  $u(i)_a = u_{(i, a)}$  belongs to  $\mathbf{P}X_i$ .

**Lemma 5.** *Let  $u' \in (\mathbb{R}^+)^{|\&_{i \in I} X_i|}$ . One has  $u' \in \mathbf{P}(\&_{i \in I} X_i)^\perp$  iff*

- $\forall i \in I \ u'(i) \in \mathbf{P}X_i^\perp$
- and  $\sum_{i \in I} \|u'(i)\|_{X_i^\perp} \leq 1$ .

The proof is quite easy. It follows that  $\&_{i \in I} X_i$  is a PCS. Moreover we can define  $\text{pr}_i \in \mathbf{Pcoh}(\&_{j \in I} X_j, X_i)$  by

$$(\text{pr}_i)_{(j, a), a'} = \begin{cases} 1 & \text{if } j = i \text{ and } a = a' \\ 0 & \text{otherwise.} \end{cases}$$

Then  $(\&_{i \in I} X_i, (\text{pr}_i)_{i \in I})$  is the cartesian product of the family  $(X_i)_{i \in I}$  in the category  $\mathbf{Pcoh}$ . The coproduct  $(\oplus_{i \in I} X_i, (\text{in}_i)_{i \in I})$  is the dual operation, so that

$$|\oplus_{i \in I} X_i| = \bigcup_{i \in I} \{i\} \times |X_i|$$

and  $u \in \mathbf{P}(\oplus_{i \in I} X_i)$  if  $\forall i \in I \ u(i) \in \mathbf{P}X_i$  and  $\sum_{i \in I} \|u(i)\|_{X_i} \leq 1$ . The injections  $\text{in}_j \in \mathbf{Pcoh}(X_j, \oplus_{i \in I} X_i)$  are given by

$$(\text{in}_i)_{a', (j, a)} = \begin{cases} 1 & \text{if } j = i \text{ and } a = a' \\ 0 & \text{otherwise.} \end{cases}$$

Given morphisms  $s_i \in \mathbf{Pcoh}(X_i, Y)$  (for each  $i \in I$ ), then the unique morphism  $s \in \mathbf{Pcoh}(\oplus_{i \in I} X_i, Y)$  is given by  $s_{(i,a),b} = (s_i)_{a,b}$  and denoted as  $\text{case}_{i \in I} s_i$  (in the binary case, we use  $\text{case}(s_1, s_2)$ ).

**3.2.6. Exponentials.** Given a set  $I$ , a *finite multiset* of elements of  $I$  is a function  $b : I \rightarrow \mathbb{N}$  whose *support*  $\text{supp}(b) = \{a \in I \mid b(a) \neq 0\}$  is finite. We use  $\mathcal{M}_{\text{fin}}(I)$  for the set of all finite multisets of elements of  $I$ . Given a finite family  $a_1, \dots, a_n$  of elements of  $I$ , we use  $[a_1, \dots, a_n]$  for the multiset  $b$  such that  $b(a) = \#\{i \mid a_i = a\}$ . We use additive notations for multiset unions:  $\sum_{i=1}^k b_i$  is the multiset  $b$  such that  $b(a) = \sum_{i=1}^k b_i(a)$ . The empty multiset is denoted as  $0$  or  $[\ ]$ . If  $k \in \mathbb{N}$ , the multiset  $kb$  maps  $a$  to  $k b(a)$ .

Let  $X$  be a PCS. Given  $u \in PX$  and  $b \in \mathcal{M}_{\text{fin}}(|X|)$ , we define  $u^b = \prod_{a \in |X|} u_a^{b(a)} \in \mathbb{R}^+$ . Then we set  $u^! = (u^b)_{b \in \mathcal{M}_{\text{fin}}(|X|)}$  and finally

$$!X = (\mathcal{M}_{\text{fin}}(|X|), \{u^! \mid u \in PX\}^{\perp\perp})$$

which is a pre-PCS.

We check quickly that  $!X$  so defined is a PCS. Let  $b = [a_1, \dots, a_n] \in \mathcal{M}_{\text{fin}}(|X|)$ . Because  $X$  is a PCS, and by Theorem 2, for each  $i = 1, \dots, n$  there is  $u(i) \in PX$  such that  $u(i)_{a_i} > 0$ . Let  $(\alpha_i)_{i=1}^n$  be a family of strictly positive real numbers such that  $\sum_{i=1}^n \alpha_i \leq 1$ . Then  $u = \sum_{i=1}^n \alpha_i u(i) \in PX$  satisfies  $u_{a_i} > 0$  for each  $i = 1, \dots, n$ . Therefore  $u_b^! = u^b > 0$ . This shows that there is  $U \in P(!X)$  such that  $U_b > 0$ .

Let now  $A \in \mathbb{R}^+$  be such that  $\forall u \in PX \forall i \in \{1, \dots, n\} u_{a_i} \leq A$ . For all  $u \in PX$  we have  $u^b \leq A^n$ . We have

$$(P(!X))^{\perp} = \{u^! \mid u \in PX\}^{\perp\perp\perp} = \{u^! \mid u \in PX\}^{\perp}.$$

Let  $t \in (\mathbb{R}^+)^{|!X|}$  be defined by  $t_c = 0$  if  $c \neq b$  and  $t_b = A^{-n} > 0$ ; we have  $t \in (P(!X))^{\perp}$ . We have exhibited an element  $t$  of  $(P(!X))^{\perp}$  such that  $t_b > 0$ . By Lemma 1 it follows that  $!X$  is a PCS.

**3.2.7. Kleisli morphisms as functions.** Let  $s \in (\mathbb{R}^+)^{|!X \multimap Y|}$ . We define a function  $\widehat{s} : PX \rightarrow \overline{\mathbb{R}^+}^{|Y|}$  as follows. Given  $u \in PX$ , we set

$$\widehat{s}(u) = s u^! = \left( \sum_{c \in |!X|} s_{c,b} u^c \right)_{b \in |Y|}.$$

**Theorem 6.** *One has  $s \in P(!X \multimap Y)$  iff, for all  $u \in PX$ , one has  $\widehat{s}(u) \in PY$ .*

This is an immediate consequence of the definition.

**Theorem 7.** *Let  $s \in \mathbf{Pcoh}(!X, Y)$ . The function  $\widehat{s}$  is Scott-continuous. Moreover, given  $s, s' \in \mathbf{Pcoh}(!X, Y)$ , one has  $s = s'$  (as matrices) iff  $\widehat{s} = \widehat{s'}$  (as functions  $PX \rightarrow PY$ ).*

This is an easy consequence of the fact that two polynomials of  $n$  variables with real coefficients are identical iff they are the same function on any open subset of  $\mathbb{R}^n$ .

So we can consider the elements of  $\mathbf{Pcoh}_!(X, Y)$  (the morphisms of the Kleisli category of the comonad  $!_-$  on the category  $\mathbf{Pcoh}$ ) as particular Scott continuous functions  $PX \rightarrow PY$  and this identification is compatible with the definition of identity maps and of composition in  $\mathbf{Pcoh}_!$ , see Section 3.1.2. Of course, not all Scott continuous function are morphisms in  $\mathbf{Pcoh}_!$ .

**Theorem 8.** *Let  $s, s' \in \mathbf{Pcoh}_!(X, Y)$  be such that  $s \leq s'$  (as elements of  $\mathbf{P}(!X \multimap Y)$ ). Then  $\forall u \in \mathbf{P}X \ \widehat{s}(u) \leq \widehat{s'}(u)$ . Let  $(s(i))_{i \in \mathbb{N}}$  be a monotone sequence of elements of  $\mathbf{Pcoh}_!(X, Y)$  and let  $s = \sup_{i \in \mathbb{N}} s(i)$ . Then  $\forall u \in \mathbf{P}X \ \widehat{s}(u) = \sup_{i \in \mathbb{N}} \widehat{s_i}(u)$ .*

The first statement is obvious. The second one results from the monotone convergence Theorem.

Given a multiset  $b \in \mathcal{M}_{\text{fin}}(I)$ , we define its *factorial*  $b! = \prod_{i \in I} b(i)!$  and its *multinomial coefficient*  $\text{mn}(b) = (\#b)!/b! \in \mathbb{N}^+$  where  $\#b = \sum_{i \in I} b(i)$  is the cardinality of  $b$ . Remember that, given an  $I$ -indexed family  $a = (a_i)_{i \in I}$  of elements of a commutative semi-ring, one has the multinomial formula

$$\left( \sum_{i \in I} a_i \right)^n = \sum_{b \in \mathcal{M}_n(I)} \text{mn}(b) a^b$$

where  $\mathcal{M}_n(I) = \{b \in \mathcal{M}_{\text{fin}}(I) \mid \#b = n\}$ .

Given  $c \in !|X|$  and  $d \in !|Y|$  we define  $\mathbf{L}(c, d)$  as the set of all multisets  $r$  in  $\mathcal{M}_{\text{fin}}(|X| \times |Y|)$  such that

$$\forall a \in |X| \sum_{b \in |Y|} r(a, b) = c(a) \quad \text{and} \quad \forall b \in |Y| \sum_{a \in |X|} r(a, b) = d(b).$$

Let  $t \in \mathbf{Pcoh}(X, Y)$ , we define  $!t \in (\mathbb{R}^+)^{!X \multimap !Y}$  by

$$(!t)_{c,d} = \sum_{r \in \mathbf{L}(c,d)} \frac{d!}{r!} t^r.$$

Observe that the coefficients in this sum are all non-negative integers.

**Lemma 9.** *For all  $u \in \mathbf{P}X$  one has  $!t u^! = (tu)^!$ .*

This results from a simple computation applying the multinomial formula.

**Theorem 10.** *For all  $t \in \mathbf{Pcoh}(X, Y)$  one has  $!t \in \mathbf{Pcoh}(!X, !Y)$  and the operation  $t \mapsto !t$  is functorial.*

Immediate consequences of Lemma 9 and Theorem 7.

**3.2.8. Description of the exponential comonad.** We equip now this functor with a structure of comonad: let  $\text{der}_X \in (\mathbb{R}^+)^{!X \multimap X}$  be given by  $(\text{der}_X)_{b,a} = \delta_{[a],b}$  (the value of the Kronecker symbol  $\delta_{i,j}$  is 1 if  $i = j$  and 0 otherwise) and  $\text{dig}_X \in (\mathbb{R}^+)^{!X \multimap !|X|}$  be given by  $(\text{dig}_X)_{b,[b_1, \dots, b_n]} = \delta_{\sum_{i=1}^n b_i, b}$ . Then we have  $\text{der}_X \in \mathbf{Pcoh}(!X, X)$  and  $\text{dig}_X \in \mathbf{Pcoh}(!X, !|X|)$  simply because

$$\widehat{\text{der}_X}(u) = u \quad \text{and} \quad \widehat{\text{dig}_X}(u) = (u^!)^!$$

for all  $u \in \mathbf{P}X$ , as easily checked. Using these equations, one also checks easily the naturality of these morphisms, and the fact that  $(!-, \text{der}, \text{dig})$  is a comonad.

As to the monoidality of this comonad, we introduce  $\mathbf{m}^0 \in (\mathbb{R}^+)^{!1 \multimap !\top}$  by  $\mathbf{m}_{*,\top}^0 = 1$  and  $\mathbf{m}_{X,Y}^2 \in (\mathbb{R}^+)^{!X \otimes !Y \multimap !(X \& Y)}$  by  $(\mathbf{m}_{X,Y}^2)_{b,c,d} = \delta_{d, 1.b + 2.c}$  where  $i.[a_1, \dots, a_n] = [(i, a_1), \dots, (i, a_n)]$ . It is easily checked that the required commutations hold (again, we refer to [18]).



**3.2.9. Fix-points in  $\mathbf{Pcoh}_!$ .** For any object  $Y$  of  $\mathbf{Pcoh}$ , a morphism  $t \in \mathbf{Pcoh}_!(Y, Y)$  defines a Scott-continuous function  $f = \hat{t} : \mathbf{P}(Y) \rightarrow \mathbf{P}(Y)$  which has a least fix-point  $\sup_{n \in \mathbb{N}} f^n(0)$ . Let  $X$  be an object of  $\mathbf{Pcoh}$  and set  $Y = !(X \multimap X) \multimap X$ . Then we have a morphism  $t = \text{cur } s \in \mathbf{Pcoh}_!(Y, Y)$  where  $s \in \mathbf{Pcoh}(Y \otimes !(X \multimap X), X)$  is defined as the following composition of morphisms in  $\mathbf{Pcoh}$ :

$$\begin{aligned} !Y \otimes !(X \multimap X) &\xrightarrow{!Y \otimes c_{!X \multimap X}} !Y \otimes !(X \multimap X) \otimes !(X \multimap X) \\ &\downarrow (\text{ev}(\text{der}_Y \otimes !(X \multimap X)))^! \otimes \text{der}_{!(X \multimap X)} \\ !X \otimes (X \multimap X) &\xrightarrow{\text{ev } \sigma} X \end{aligned}$$

Then  $\hat{t}$  is a Scott continuous function  $\mathbf{P}Y \rightarrow \mathbf{P}Y$  whose least fix-point is  $\text{fix}$ , considered as a morphism  $\text{fix} \in \mathbf{Pcoh}_!(X \multimap X, X)$ , satisfies  $\widehat{\text{fix}}(u) = \sup_{n=0}^{\infty} \hat{u}^n(0)$ .

**3.2.10. The partially ordered class of probabilistic coherence spaces.** We define the category  $\mathbf{Pcoh}_{\subseteq}$ . This category is actually a partially ordered class whose objects are those of  $\mathbf{Pcoh}$ . The order relation, denoted as  $\subseteq$ , is defined as follows:  $X \subseteq Y$  if  $|X| \subseteq |Y|$  and the matrices  $\eta_{X,Y}^+$  and  $\eta_{X,Y}^-$  defined, for  $a \in |X|$  and  $b \in |Y|$ , by  $(\eta_{X,Y}^+)_{a,b} = (\eta_{X,Y}^-)_{b,a} = \delta_{a,b}$  satisfy  $\eta_{X,Y}^+ \in \mathbf{Pcoh}(X, Y)$  and  $\eta_{X,Y}^- \in \mathbf{Pcoh}(Y, X)$ . In other words: given  $u \in \mathbf{P}X$ , the element  $\eta_{X,Y}^+ u$  of  $(\mathbb{R}^+)^{|Y|}$  obtained by extending  $u$  with 0's outside  $|X|$  belongs to  $\mathbf{P}Y$ . And conversely, given  $v \in \mathbf{P}Y$ , the element  $\eta_{X,Y}^- v$  of  $(\mathbb{R}^+)^{|X|}$  obtained by restricting  $v$  to  $|X|$  belongs to  $\mathbf{P}X$ . Considering  $\mathbf{Pcoh}_{\subseteq}$  as a category,  $\eta_{X,Y}$  is a notation for the unique element of  $\mathbf{Pcoh}_{\subseteq}(X, Y)$  when  $X \subseteq Y$ , in accordance with the notations of Paragraph 3.1.3.

**Lemma 11.** *If  $X \subseteq Y$  then  $X^{\perp} \subseteq Y^{\perp}$ ,  $\eta_{X^{\perp}, Y^{\perp}}^+ = (\eta_{X,Y}^-)^{\perp}$  and  $\eta_{X^{\perp}, Y^{\perp}}^- = (\eta_{X,Y}^+)^{\perp}$ .*

The proof is a straightforward verification.

We contend that  $\mathbf{Pcoh}_{\subseteq}$  is directed co-complete. Let  $(X_{\gamma})_{\gamma \in \Gamma}$  be a countable directed family in  $\mathbf{Pcoh}_{\subseteq}$  (so  $\Gamma$  is a countable directed poset and  $\gamma \leq \gamma' \Rightarrow X_{\gamma} \subseteq X_{\gamma'}$ ), we have to check that this family has a least upper bound  $X$ . We set  $|X| = \bigcup_{\gamma \in \Gamma} |X_{\gamma}|$  and  $\mathbf{P}X = \{w \in (\mathbb{R}^+)^{|X|} \mid \forall \gamma \in \Gamma \ \eta_{X,Y}^- w \in \mathbf{P}X_{\gamma}\}$ . This defines an object of  $\mathbf{Pcoh}$  which satisfies  $\mathbf{P}X = \{\eta_{X_{\gamma}, X}^+ u \mid \gamma \in \Gamma \text{ and } u \in \mathbf{P}X_{\gamma}\}^{\perp\perp}$  and is therefore the lub of the family  $(X_{\gamma})_{\gamma \in \Gamma}$  in  $\mathbf{Pcoh}_{\subseteq}$ . This object  $X$  is denoted  $\bigcup_{\gamma \in \Gamma} X_{\gamma}$ . One checks easily that  $(\bigcup_{\gamma \in \Gamma} X_{\gamma})^{\perp} = \bigcup_{\gamma \in \Gamma} X_{\gamma}^{\perp}$ .

Then the functor  $\mathbf{E} : \mathbf{Pcoh}_{\subseteq} \rightarrow \mathbf{Pcoh}$  defined by  $\mathbf{E}(X) = X$  and  $\mathbf{E}(\eta_{X,Y}) = \eta_{X,Y}^+$  is continuous: given a directed family  $(X_{\gamma})_{\gamma \in \Gamma}$  whose lub is  $X$  and given a collection of morphisms  $t_{\gamma} \in \mathbf{Pcoh}(X_{\gamma}, Y)$  such that  $t_{\gamma'} \eta_{X_{\gamma}, X_{\gamma'}}^+ = t_{\gamma}$  for any  $\gamma, \gamma' \in \Gamma$  such that  $\gamma \leq \gamma'$ , there is exactly one morphism  $t \in \mathbf{Pcoh}(X, Y)$  such that  $t \eta_{X_{\gamma}, X}^+ = t_{\gamma}$  for each  $\gamma \in \Gamma$ . Given  $a \in |X|$  and  $b \in |Y|$ ,  $t_{a,b} = (t_{\gamma})_{a,b}$  for any  $\gamma$  such that  $a \in |X_{\gamma}|$  (our hypothesis on the  $t_{\gamma}$ 's means that  $(t_{\gamma})_{a,b}$  does not depend on the choice of  $\gamma$ ).

All the operations of Linear Logic define monotone continuous functionals on  $\mathbf{Pcoh}_{\subseteq}$  which moreover commute with the functor  $\mathbf{F}$ . This means for instance that if  $X \subseteq Y$  then  $!X \subseteq !Y$ ,  $\eta_{!X, !Y}^+ = !(\eta_{X,Y}^+)$ ,  $\eta_{!X, !Y}^- = !(\eta_{X,Y}^-)$  and  $!(\bigcup_{\gamma \in \Gamma} X_{\gamma}) = \bigcup_{\gamma \in \Gamma} !X_{\gamma}$  and similarly for  $\otimes$  and  $\oplus$ . As a consequence, and as a consequence of Lemma 11, if  $X_i \subseteq Y_i$  for  $i = 1, 2$  then  $X_1 \multimap X_2 \subseteq Y_1 \multimap Y_2$ ,  $\eta_{X_1 \multimap X_2, Y_1 \multimap Y_2}^+ = \eta_{X_1, Y_1}^- \multimap \eta_{X_2, Y_2}^+$  and  $\eta_{X_1 \multimap X_2, Y_1 \multimap Y_2}^- = \eta_{X_1, Y_1}^+ \multimap \eta_{X_2, Y_2}^-$  and  $\multimap$  commutes with directed colimits in  $\mathbf{Pcoh}_{\subseteq}$ .

This notion of inclusion on probabilistic coherence spaces extends to coalgebras as outlined in Section 3.1.3 (again, we refer to [4] for more details). We describe briefly this notion of inclusion in the present concrete setting.

Let  $P$  and  $Q$  be object of  $\mathbf{Pcoh}^!$ , we have  $P \subseteq Q$  in  $\mathbf{Pcoh}^!_{\subseteq}$  if  $\underline{P} \subseteq \underline{Q}$  and  $h_Q \eta_{\underline{P}, \underline{Q}}^+ = !(\eta_{\underline{P}, \underline{Q}}^+) h_P$ . The lub of a directed family  $(P_\gamma)_{\gamma \in \Gamma}$  of coalgebras (for this notion of substructure) is the coalgebra  $P = \bigcup_{\gamma \in \Gamma} P_\gamma$  defined by  $\underline{P} = \bigcup_{\gamma \in \Gamma} \underline{P}_\gamma$  and  $h_P$  is characterized by the equation  $h_P \eta_{\underline{P}_\gamma, \underline{P}}^+ = !\eta_{\underline{P}_\gamma, \underline{P}}^+ h_{P_\gamma}$  which holds for each  $\gamma \in \Gamma$ .

As outlined in Section 3.1.4, this allows to interpret any type  $\sigma$  as an object  $[\sigma]$  of  $\mathbf{Pcoh}$  and any positive type  $\varphi$  as an object  $[\varphi]^!$  such that  $[\varphi]^! = [\varphi]$ , in such a way that the coalgebras  $[\mathbf{Fix} \zeta \cdot \varphi]^!$  and  $[\varphi [\mathbf{Fix} \zeta \cdot \varphi / \varphi]]^!$  are exactly the same. We use  $h_\varphi$  for  $h_{[\varphi]^!}$ .

**3.2.11. Dense coalgebras.** Let  $P$  be an object of  $\mathbf{Pcoh}^!$ , so that  $P = (\underline{P}, h_P)$  where  $\underline{P}$  is a probabilistic coherence space and  $h_P \in \mathbf{Pcoh}(\underline{P}, !\underline{P})$  satisfies  $\text{dig}_{\underline{P}} h_P = !h_P$ . Given coalgebras  $P$  and  $Q$ , a morphism  $t \in \mathbf{Pcoh}(\underline{P}, \underline{Q})$  is coalgebraic (that is  $t \in \mathbf{Pcoh}^!(P, Q)$ ) if  $h_Q t = !t h_P$ . In particular, we say that  $u \in P(\underline{P})$  is coalgebraic if, considered as a morphism from 1 to  $\underline{P}$ ,  $u$  belongs to  $\mathbf{Pcoh}^!(1, P)$ . This means that  $u^! = h_P u$ .

**Definition 12.** Given an object  $P$  of  $\mathbf{Pcoh}^!$ , we use  $P^!(P)$  for the set of coalgebraic elements of  $P(\underline{P})$ .

The following lemma is useful in the sequel and holds in any model of Linear Logic.

**Lemma 13.** Let  $X$  be a probabilistic coherence space, one has  $P^!(!X) = \{u^! \mid u \in PX\}$ . Let  $P_1$  and  $P_2$  be objects of  $\mathbf{Pcoh}^!$ .

$P_1 \otimes P_2$  is the cartesian product of  $P_1$  and  $P_2$  in  $\mathbf{Pcoh}^!$ . The function  $P^!(P_1) \times P^!(P_2) \rightarrow P^!(P_1 \otimes P_2)$  which maps  $(u, v)$  to  $u \otimes v$  is a bijection. The projections  $\text{pr}_i^\otimes \in \mathbf{Pcoh}^!(P_1 \otimes P_2, P_i)$  are characterized by  $\text{pr}_i^\otimes(u_1 \otimes u_2) = u_i$ .

The function  $\{1\} \times P^!(P_1) \cup \{2\} \times P^!(P_2) \rightarrow P^!(P_1 \oplus P_2)$  which maps  $(i, u)$  to  $\text{in}_i(u)$  is a bijection. This injection has a left inverse  $\text{pr}_i \in \mathbf{Pcoh}(\underline{P_1} \oplus \underline{P_2}, \underline{P_i})$  defined by  $(\text{pr}_i)_{(j,a),b} = \delta_{i,j} \delta_{a,b}$ , which is not a coalgebra morphism in general.

*Proof.* Let  $v \in P^!(!X)$ , we have  $v^! = h_{!X} v = \text{dig}_X v$  hence  $(\text{der}_X v)^! = !\text{der}_X v^! = !\text{der}_X \text{dig}_X v = v$ . The other properties result from the fact that the Eilenberg-Moore category  $\mathbf{Pcoh}^!$  is cartesian and co-cartesian with  $\otimes$  and  $\oplus$  as product and co-product, see [18] for more details.  $\square$

Because of these properties we write sometimes  $(u_1, u_2)$  instead of  $u_1 \otimes u_2$  when  $u_i \in P^!P_i$  for  $i = 1, 2$ .

**Definition 14.** An object  $P$  of  $\mathbf{Pcoh}^!$  is *dense* if, for any object  $Y$  of  $\mathbf{Pcoh}$  and any two morphisms  $t, t' \in \mathbf{Pcoh}(\underline{P}, Y)$ , if  $t u = t' u$  for all  $u \in P^!(P)$ , then  $t = t'$ .

**Theorem 15.** For any probabilistic coherence space  $X$ ,  $!X$  is a dense coalgebra. If  $P_1$  and  $P_2$  are dense coalgebras then  $P_1 \otimes P_2$  and  $P_1 \oplus P_2$  are dense. The colimit of a directed family of dense coalgebras is dense.

*Proof.* Let  $X$  be an object of  $\mathbf{Pcoh}$ , one has  $P^!(!X) = \{u^! \mid u \in PX\}$  by Lemma 13. It follows that  $!X$  is a dense coalgebra by Theorem 7. Assume that  $P_1$  and  $P_2$  are dense coalgebras. Let  $t, t' \in \mathbf{Pcoh}(\underline{P_1} \otimes \underline{P_2}, Y)$  be such that  $t w = t' w$  for all  $w \in P^!(P_1 \otimes P_2)$ . We

have  $\text{cur}(t), \text{cur}(t') \in \mathbf{Pcoh}(\underline{P}_1, \underline{P}_2 \multimap Y)$  so, using the density of  $P_1$ , it suffices to prove that  $\text{cur}(t)u_1 = \text{cur}(t')u_1$  for each  $u_1 \in P^!(P_1)$ . So let  $u_1 \in P^!(P_1)$  and let  $s = \text{cur}(t)u_1$  and  $s' = \text{cur}(t')u_1$ . Let  $u_2 \in P^!(P_2)$ , we have  $su_2 = t(u_1 \otimes u_2) = t'(u_1 \otimes u_2) = s'u_2$  since  $u_1 \otimes u_2 \in P^!(P_1 \otimes P_2)$  and therefore  $s = s'$  since  $P_2$  is dense. Let now  $t, t' \in \mathbf{Pcoh}(\underline{P}_1 \oplus \underline{P}_2, Y)$  be such that  $tw = t'w$  for all  $w \in P^!(P_1 \oplus P_2)$ . To prove that  $t = t'$ , it suffices to prove that  $t \text{in}_i = t' \text{in}_i$  for  $i = 1, 2$ . Since  $P_i$  is dense, it suffices to prove that  $t \text{in}_i u = t' \text{in}_i u$  for each  $u \in P^!(P_i)$  which follows from the fact that  $\text{in}_I u \in P^!P_i$ . Last let  $(P_\gamma)_{\gamma \in \Gamma}$  be a directed family of dense coalgebras (in  $\mathbf{Pcoh}_{\subseteq}^!$ ) and let  $P = \bigcup_{\gamma \in \Gamma} P_\gamma$ , and let  $t, t' \in \mathbf{Pcoh}^!(\underline{P}, Y)$  be such that  $tw = t'w$  for all  $w \in P^!(P)$ . It suffices to prove that, for each  $\gamma \in \Gamma$ , one has  $t \eta_{\underline{P}_\gamma, \underline{P}}^+ = t' \eta_{\underline{P}_\gamma, \underline{P}}^+$  and this results from the fact that  $P_\gamma$  is dense and  $\eta_{\underline{P}_\gamma, \underline{P}}^+$  is a coalgebra morphisms (and therefore maps  $P^!(P_\gamma)$  to  $P^!(P)$ ).  $\square$

The sub-category  $\mathbf{Pcoh}^!$  of dense coalgebras is cartesian and co-cartesian and is well-pointed by Theorem 15. We use  $\mathbf{Pcoh}_{\text{den}}^!$  for this sub-category.

**3.2.12. Interpreting types and terms in  $\mathbf{Pcoh}$ .** Given a type  $\sigma$  with free type variables contained in the repetition-free list  $\vec{\zeta}$ , and given a sequence  $\vec{P}$  of length  $n$  of objects of  $\mathbf{Pcoh}^!$ , we define  $[\sigma]_{\vec{\zeta}}(\vec{P})$  as an object of  $\mathbf{Pcoh}$  and when  $\varphi$  is a positive type (whose free variables are contained in  $\vec{\zeta}$ ) we define  $[\varphi]_{\vec{\zeta}}^!(\vec{P})$  as an object of  $\mathbf{Pcoh}^!$ . These operations are continuous and their definition follows the general pattern described in Section 3.1.4.

**Theorem 16.** *For any closed positive type  $\varphi$ , the coalgebra  $[\varphi]^!$  is dense.*

This is an immediate consequence of Theorem 15.

Then  $o = \mathbf{1} \oplus \mathbf{1}$  satisfies  $\|o\| = \{(1, *), (2, *)\}$  and  $u \in (\mathbb{R}^+)^{\|o\|}$  satisfies  $u \in P[o]$  iff  $u_{(1,*)} + u_{(2,*)} \leq 1$ . The coalgebraic structure of this object is given by

$$(\text{h}_o)_{(i,*),[(j_1,*),\dots,(j_k,*)]} = \begin{cases} 1 & \text{if } j = j_1 = \dots = j_k \\ 0 & \text{otherwise.} \end{cases}$$

The object  $\mathbf{N} = [u]$  satisfies  $\mathbf{N} = \mathbf{1} \oplus \mathbf{N}$  so that  $|\mathbf{N}| = \{(1, *), (2, (1, *)), (2, (2, (1, *))), \dots\}$  and we use  $\bar{n}$  for the element of  $|\mathbf{N}|$  which has  $n$  occurrences of 2. Given  $u \in (\mathbb{R}^+)^{|\mathbf{N}|}$ , we use  $l(u)$  for the element of  $(\mathbb{R}^+)^{|\mathbf{N}|}$  defined by  $l(u)_{\bar{n}} = u_{\overline{n+1}}$ . By definition of  $\mathbf{N}$ , we have  $u \in \mathbf{PN}$  iff  $u_{\bar{0}} + \|u\|_{\mathbf{N}} \leq 1$ , and then  $\|u\|_{\mathbf{N}} = u_{\bar{0}} + \|l(u)\|_{\mathbf{N}}$ . It follows that  $u \in \mathbf{PN}$  iff  $\sum_{n=0}^{\infty} u_{\bar{n}} \leq 1$  and of course  $\|u\|_{\mathbf{N}} = \sum_{n=0}^{\infty} u_{\bar{n}}$ . Then the coalgebraic structure  $\text{h}_l$  is defined exactly as  $\text{h}_o$  above. In the sequel, we identify  $|\mathbf{N}|$  with  $\mathbf{N}$ .

Given a typing context  $\mathcal{P} = (x_1 : \varphi_1, \dots, x_k : \varphi_k)$ , a type  $\sigma$  and a term  $M$  such that  $\mathcal{P} \vdash M : \sigma$ ,  $M$  is interpreted as a morphism  $[M]^{\mathcal{P}} \in \mathbf{Pcoh}([\mathcal{P}], [\sigma])$ . For all constructs of the language but probabilistic choice, this interpretation uses the generic structures of the model described in Section 3.1, the description of this interpretation can be found in [4]. We set  $[\text{coin}(p)] = pe_{(1,*)} + (1-p)e_{(2,*)}$ .

If  $x_1 : \varphi_1, \dots, x_k : \varphi_k \vdash M : \sigma$ , the morphism  $[M]^{\mathcal{P}}$  is completely characterized by its values on  $(u_1, \dots, u_k) \in P^!([\mathcal{P}]^!)$ . We describe now the interpretation of terms using this observation.

- $[()] = 1 \in P1 = [0, 1]$ .
- $[x_i]^{\mathcal{P}}(u_1, \dots, u_k) = u_i$ .
- $[N^!]^{\mathcal{P}}(u_1, \dots, u_k) = ([N]^{\mathcal{P}}(u_1, \dots, u_k))^!$ .

- $[(M_1, M_2)]^{\mathcal{P}}(u_1, \dots, u_k) = [M_1]^{\mathcal{P}}(u_1, \dots, u_k) \otimes [M_2]^{\mathcal{P}}(u_1, \dots, u_k)$ .
- $[\text{in}_i N]^{\mathcal{P}}(u_1, \dots, u_k) = \text{in}_i([N]^{\mathcal{P}}(u_1, \dots, u_k))$ .
- $[\text{der}(N)]_{\mathcal{P}}(u_1, \dots, u_k) = \text{der}_{[\sigma]}([N]^{\mathcal{P}}(u_1, \dots, u_k))$ , assuming that  $\mathcal{P} \vdash N : !\sigma$ .
- If  $\mathcal{P} \vdash N : \varphi \multimap \sigma$  and  $\mathcal{P} \vdash R : \varphi$  then  $[N]^{\mathcal{P}}(u_1, \dots, u_k) \in \mathbf{P}([\varphi] \multimap [\sigma])$ , and  $[R]^{\mathcal{P}}(u_1, \dots, u_k) \in \mathbf{P}([\varphi])$  and using the application of a matrix to a vector we have  $[\langle N \rangle R]^{\mathcal{P}}(u_1, \dots, u_k) = [N]^{\mathcal{P}}(u_1, \dots, u_k) [R]^{\mathcal{P}}(u_1, \dots, u_k)$ .
- If  $\mathcal{P}, x : \varphi \vdash N : \sigma$  then  $[\lambda x^{\varphi} N]^{\mathcal{P}}(u_1, \dots, u_k) \in \mathbf{P}([\varphi] \multimap [\sigma])$  is completely described by the fact that, for all  $u \in \mathbf{P}^!([\varphi]^!)$ , one has  $[\lambda x^{\varphi} N]^{\mathcal{P}}(u_1, \dots, u_k) u = [N]^{\mathcal{P}, x:\varphi}(u_1, \dots, u_k, u)$ . This is a complete characterization of this interpretation by Theorem 16.
- If  $\mathcal{P} \vdash N : \varphi_1 \oplus \varphi_2$  and  $\mathcal{P}, y_i : \varphi_i \vdash R_i : \sigma$  for  $i = 1, 2$ , then  $[\text{case}(N, y_1 \cdot R_1, y_2 \cdot R_2)]^{\mathcal{P}}(u_1, \dots, u_k) = [R_1]_{\mathcal{P}, y_1:\varphi_1}(u_1, \dots, u_k, \text{pr}_1([N]^{\mathcal{P}}(u_1, \dots, u_k))) + [R_2]_{\mathcal{P}, y_2:\varphi_2}(u_1, \dots, u_k, \text{pr}_2([N]^{\mathcal{P}}(u_1, \dots, u_k)))$  where  $\text{pr}_i \in \mathbf{Pcoh}(P_1 \oplus P_2, P_i)$  is the  $i$ th “projection” introduced in 3.2.11, left inverse for  $\text{in}_i$ .
- If  $\mathcal{P}, x : !\sigma \vdash N : \sigma$  then  $[N]_{\mathcal{P}, x:!\sigma} \in \mathbf{Pcoh}([\mathcal{P}] \otimes ![\sigma], [\sigma])$  and  $[\text{fix } x^{!\sigma} N]^{\mathcal{P}}(u_1, \dots, u_k) = \sup_{n=0}^{\infty} f^n(0)$  where  $f : \mathbf{P}[\sigma] \rightarrow \mathbf{P}[\sigma]$  is the Scott-continuous function given by  $f(u) = [N]_{\mathcal{P}, x:!\sigma}(u_1, \dots, u_k, u^!)$ .
- If  $\mathcal{P} \vdash N : \psi [\mathbf{Fix} \zeta \cdot \psi / \zeta]$  then  $[\text{fold}(N)]^{\mathcal{P}} = [N]^{\mathcal{P}}$  which makes sense since  $[\psi [\mathbf{Fix} \zeta \cdot \psi / \zeta]] = [\mathbf{Fix} \zeta \cdot \psi]$ .
- If  $\mathcal{P} \vdash N : \mathbf{Fix} \zeta \cdot \psi$  then  $[\text{unfold}(N)]^{\mathcal{P}} = [N]^{\mathcal{P}}$ .

**Theorem 17** (Soundness). *If  $M$  satisfies  $\mathcal{P} \vdash M : \sigma$  then*

$$[M]_{\mathcal{P}} = \sum_{\mathcal{P} \vdash M' : \sigma} \text{Red}_{M, M'} [M']_{\mathcal{P}}$$

The proof is a straightforward verification.

**3.3. Examples of term interpretations.** We give the interpretation of terms that we gave as examples in Subsection 2.3.

- $[\Omega^{\sigma}] = [\text{fix } x^{!\sigma} \text{der}(x)] = 0$
- $[\mathbf{t}] = e_{(1,*)}$  and  $[\mathbf{f}] = e_{(2,*)}$
- $[\underline{n}] = \bar{n}$  for  $n \in \mathbb{N}$
- $[\text{suc}(M)]_{n+1}^{\mathcal{P}}(u_1, \dots, u_k) = [M]_n^{\mathcal{P}}(u_1, \dots, u_k)$
- $[\text{if}(M, N_1, (x)N_2)]^{\mathcal{P}}(u_1, \dots, u_k) = [M]_0^{\mathcal{P}}(u_1, \dots, u_k) [N_1]^{\mathcal{P}}(u_1, \dots, u_k) + \sum_{n=0}^{\infty} [M]_{n+1}^{\mathcal{P}}(u_1, \dots, u_k) [N_2]^{\mathcal{P}}(u_1, \dots, u_k)(\bar{n})$
- $[\text{dice}_p(M_1, M_2)]^{\mathcal{P}}(u_1, \dots, u_k) = p [M_1]^{\mathcal{P}}(u_1, \dots, u_k) + (1-p) [M_2]^{\mathcal{P}}(u_1, \dots, u_k)$
- $[\text{ran}(\vec{p})] = \sum_{i=1}^n p_i e_{\vec{i}}$
- $[\langle \text{prob}_{\ell} \rangle M]^{\mathcal{P}}(u_1, \dots, u_k) = [M]^{\mathcal{P}}(u_1, \dots, u_k)_{\ell} e_{*}$
- $[M_1 \cdot N]^{\mathcal{P}}(u_1, \dots, u_k) = [M_1]^{\mathcal{P}}(u_1, \dots, u_k) [N]^{\mathcal{P}}(u_1, \dots, u_k)$
- $[M_1 \wedge \dots \wedge M_{\ell}]^{\mathcal{P}}(u_1, \dots, u_k) = \prod_{i=1}^{\ell} [M_i]^{\mathcal{P}}(u_1, \dots, u_k)$
- $[\sum_{i=1}^k N_i \cdot M_i]^{\mathcal{P}}(u_1, \dots, u_k) = \sum_{i=1}^k [N_i]^{\mathcal{P}}(u_1, \dots, u_k) \cdot [M_i]^{\mathcal{P}}(u_1, \dots, u_k)$

$$\begin{aligned}
& \bullet \forall u \in \mathbf{P}([l]), [\text{ext}(l, r)](u) = \sum_{i=l}^r u_i e_i^- \text{ and} \\
& [\text{win}_\ell(\vec{n})](u) = \sum_{i=n_1+\dots+n_{\ell-1}+1}^{n_1+\dots+n_\ell} u_i e_i^-
\end{aligned}$$

**3.4. Adequacy.** Our goal is to prove an Adequacy Theorem stating that, for any closed term  $M$  such that  $\vdash M : \mathbf{1}$ , the probability that  $M$  reduces to  $()$  is larger than  $[M] \in \mathbf{P}[\mathbf{1}] \simeq [0, 1]$ . In spite of its very simple statement, the proof of this property is rather long mainly because we have to deal with the recursive type definitions allowed by our syntax. As usual, the proof is based on the definition of a logical relations between terms and elements of the model (more precisely, given any type  $\sigma$ , we have to define a relation between closed terms of types  $\sigma$  and elements of  $\mathbf{P}[\sigma]$ ; let us call such a relation a  $\sigma$ -relation).

Since we have no positivity restrictions on the occurrence of type variables wrt. which recursive types are defined, we use a very powerful technique introduced in [19] for defining this logical relation. Indeed a type variable  $\zeta$  can have positive and negative occurrences in a positive type  $\varphi$ , consider for instance the case  $\varphi = !(\zeta \multimap \zeta)$ . To define the logical relation associated with  $\mathbf{Fix} \zeta \cdot \varphi$ , we have to find a fix-point for the operation which maps a  $(\mathbf{Fix} \zeta \cdot \varphi)$ -relation  $\mathcal{R}$  to the relation  $\Phi(\mathcal{R}) = !(\mathcal{R} \multimap \mathcal{R})$  (which can be defined using  $\mathcal{R}$  as a “logical relation” in a fairly standard way). Relations are naturally ordered by inclusion, and this strongly suggests to define the above fix-point using this order relation by e.g. Tarski’s Fix-point Theorem. The problem however is that  $\Phi$  is neither a monotone nor an anti-monotone operation on relations, due to the fact that  $\zeta$  has a positive and a negative occurrence in  $\varphi$ .

It is here that Pitts’s trick comes in: we replace the relations  $\mathcal{R}$  with pairs of relations  $\mathcal{R} = (\mathcal{R}^-, \mathcal{R}^+)$  ordered as follows:  $\mathcal{R} \sqsubseteq \mathcal{S}$  if  $\mathcal{R}^+ \subseteq \mathcal{S}^+$  and  $\mathcal{S}^- \subseteq \mathcal{R}^-$ . Then we define accordingly  $\Phi(\mathcal{R})$  as a pair of relations by  $\Phi(\mathcal{R})^- = !(\mathcal{R}^+ \multimap \mathcal{R}^-)$  and  $\Phi(\mathcal{R})^+ = !(\mathcal{R}^- \multimap \mathcal{R}^+)$ . Now the operation  $\Phi$  is monotone wrt. the  $\sqsubseteq$  relation and it becomes possible to apply Tarski’s Fix-point Theorem to  $\Phi$  and get a pair of relations  $\mathcal{R}$  such that  $\mathcal{R} = \Phi(\mathcal{R})$ . The next step consists in proving that  $\mathcal{R}^- = \mathcal{R}^+$ . This is obtained by means of an analysis of the definition of the interpretation of fixpoints of types as colimits in the category  $\mathbf{Pcoh}_\subseteq$ . One is finally in position of proving a fairly standard “Logical Relation Lemma” from which adequacy follows straightforwardly.

In this short description of our adequacy proof, many technicalities have obviously been hidden, the most important one being that values are handled in a special way so that we actually consider two kinds of pairs of relations. Also, a kind of “biorthogonality closure” plays an essential role in the handling of positive types, no surprise for the readers acquainted with Linear Logic, see for instance the proof of normalization in [10].

**3.4.1. Pairs of relations and basic operations.** Given a *closed* type  $\sigma$ , we define  $\text{Rel}(\sigma)$  as the set of all pairs of relations  $\mathcal{R} = (\mathcal{R}^-, \mathcal{R}^+)$  such that, for  $\varepsilon \in \{+, -\}$ , each element of  $\mathcal{R}^\varepsilon$  is a pair  $(M, u)$  where  $\vdash M : \sigma$  and  $u \in \mathbf{P}[\sigma]$ . For a *closed* positive type  $\varphi$ , we also define  $\text{Rel}^\vee(\varphi)$  as the set of all pairs of relations  $\mathcal{V} = (\mathcal{V}^-, \mathcal{V}^+)$  such that, for  $\varepsilon \in \{+, -\}$ , each element of  $\mathcal{V}^\varepsilon$  is a pair  $(V, v)$  where  $\vdash V : \varphi$  is a value and  $v \in \mathbf{P}^![\varphi]$ .

Given  $\mathcal{R}, \mathcal{S} \in \text{Rel}(\sigma)$ , we write  $\mathcal{R} \sqsubseteq \mathcal{S}$  if  $\mathcal{R}^+ \subseteq \mathcal{S}^+$  and  $\mathcal{S}^- \subseteq \mathcal{R}^-$ . We define similarly  $\mathcal{V} \sqsubseteq \mathcal{W}$  for  $\mathcal{V}, \mathcal{W} \in \text{Rel}^\vee(\varphi)$ . Then  $\text{Rel}(\sigma)$  is a complete meet-lattice, the infimum of a

- Let  $\mathcal{R} \in \text{Rel}(\sigma)$ , we define  $!\mathcal{R} \in \text{Rel}^\vee(!\sigma)$  by:  $!\mathcal{R}^\varepsilon = \{(M^!, u^!) \mid (M, u) \in \mathcal{R}^\varepsilon\}$  for  $\varepsilon \in \{-, +\}$ .
- Let  $\mathcal{V}_i \in \text{Rel}^\vee(\varphi_i)$  for  $i = 1, 2$ . We define  $(\mathcal{V}_1 \otimes \mathcal{V}_2)^\varepsilon = \{((V_1, V_2), v_1 \otimes v_2) \mid (V_i, v_i) \in \mathcal{V}_i^\varepsilon\}$  for  $\varepsilon \in \{-, +\}$ , so that  $\mathcal{V}_1 \otimes \mathcal{V}_2 \in \text{Rel}^\vee(\varphi_1 \otimes \varphi_2)$ .
- Let  $\mathcal{V}_i \in \text{Rel}^\vee(\varphi_i)$  for  $i = 1, 2$ . We define  $(\mathcal{V}_1 \oplus \mathcal{V}_2)^\varepsilon = \{(\text{in}_i V, \text{in}_i(v)) \mid i \in \{1, 2\} \text{ and } (V, v) \in \mathcal{V}_i^\varepsilon\}$  for  $\varepsilon \in \{-, +\}$ , so that  $\mathcal{V}_1 \oplus \mathcal{V}_2 \in \text{Rel}^\vee(\varphi_1 \oplus \varphi_2)$ .
- Let  $\mathcal{V} \in \text{Rel}^\vee(\varphi)$  and  $\mathcal{R} \in \text{Rel}(\sigma)$ . We define  $\mathcal{V} \multimap \mathcal{R} \in \text{Rel}(\varphi \multimap \sigma)$  as follows:  $(\mathcal{V} \multimap \mathcal{R})^\varepsilon = \{(M, u) \mid \vdash M : \varphi \multimap \sigma, u \in \text{P}[\varphi \multimap \sigma] \text{ and } \forall (V, v) \in \mathcal{V}^{-\varepsilon} (\langle M \rangle V, u(v)) \in \mathcal{R}^\varepsilon\}$ .
- Last, given  $\mathcal{V} \in \text{Rel}^\vee(\varphi)$ , we define  $\overline{\mathcal{V}} \in \text{Rel}(\varphi)$  as follows:  $\overline{\mathcal{V}}^\varepsilon$  is the set of all  $(M, u)$  such that  $\vdash M : \varphi, u \in \text{P}[\varphi]$  and, for all  $(T, t) \in (\mathcal{V} \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ , one has  $(\langle T \rangle M, t(u)) \in \mathcal{R}(\mathbf{1})$ .

Figure 3: Logical operations for pairs of relations

collection  $(\mathcal{R}_i)_{i \in I}$  being  $\prod_{i \in I} \mathcal{R}_i = (\bigcup_{i \in I} \mathcal{R}_i^-, \bigcap_{i \in I} \mathcal{R}_i^+)$ . The same holds of course for  $\text{Rel}^\vee(\varphi)$  and we use the same notations.

We define  $\mathcal{R}(\mathbf{1})$  as the set of all pairs  $(M, p)$  such that  $\vdash M : \mathbf{1}, p \in [0, 1]$  and  $\text{Red}_{M, ()}^\infty \geq p$ .

We define in Figure 3 logical operations on these pairs of relations. The last one is the aforementioned biorthogonality closure operation on pairs of relations.

Observe that all these operations are monotone wrt.  $\sqsubseteq$ . For instance  $\mathcal{V} \sqsubseteq \mathcal{W} \wedge \mathcal{R} \sqsubseteq \mathcal{S} \Rightarrow (\mathcal{V} \multimap \mathcal{R}) \sqsubseteq (\mathcal{W} \multimap \mathcal{S})$ , and  $\mathcal{V} \sqsubseteq \mathcal{W} \Rightarrow \overline{\mathcal{V}} \sqsubseteq \overline{\mathcal{W}}$ .

**3.4.2. Fixpoints of pairs of relations.** To deal with fix-point types  $\text{Fix } \zeta \cdot \varphi$ , we need to consider types parameterized by relations as follows.

Let  $\sigma$  be a type and let  $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$  be a list of type variables without repetitions and which contains all free variables of  $\sigma$ . For all list  $\vec{\varphi} = (\varphi_1, \dots, \varphi_n)$  of *closed* positive types, we define

$$\mathcal{R}(\sigma)_{\vec{\zeta}} : \prod_{i=1}^n \text{Rel}^\vee(\varphi_i) \rightarrow \text{Rel}(\sigma \left[ \vec{\varphi} / \vec{\zeta} \right]).$$

Let also  $\varphi$  be a positive type whose free variables are contained in  $\vec{\zeta}$ , we define

$$\mathcal{V}(\varphi)_{\vec{\zeta}} : \prod_{i=1}^n \text{Rel}^\vee(\varphi_i) \rightarrow \text{Rel}^\vee(\varphi \left[ \vec{\varphi} / \vec{\zeta} \right]).$$

The definition is by simultaneous induction on  $\sigma$  and  $\varphi$ . All cases but one consist in applying straightforwardly the above defined logical operations on pairs of relations, for instance

$$\mathcal{R}(\varphi \multimap \tau)_{\vec{\zeta}}(\vec{\mathcal{V}}) = \mathcal{V}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}}) \multimap \mathcal{R}(\tau)_{\vec{\zeta}}(\vec{\mathcal{V}}) \quad \text{and} \quad \mathcal{R}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}}) = \overline{\mathcal{V}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}})}.$$

We are left with the case of recursive definitions of types, so assume that  $\varphi = \text{Fix } \zeta \cdot \psi$ . Let  $\vec{\varphi} = (\varphi_1, \dots, \varphi_n)$  be a list of closed positive types and let  $\vec{\mathcal{V}} \in \prod_{i=1}^n \text{Rel}^\vee(\varphi_i)$ , we set

$$\mathcal{V}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}}) = \bigsqcap \{ \mathcal{V} \in \text{Rel}^\vee(\varphi \left[ \vec{\varphi} / \vec{\zeta} \right]) \mid \text{fold}(\mathcal{V}(\psi)_{\vec{\zeta}, \zeta}(\vec{\mathcal{V}}, \mathcal{V})) \sqsubseteq \mathcal{V} \} \quad (3.1)$$

where we use the following notation: given  $\mathcal{W} \in \text{Rel}^v(\psi \left[ \vec{\varphi}/\vec{\zeta}, \varphi \left[ \vec{\varphi}/\vec{\zeta} \right] / \zeta \right)$ ,  $\text{fold}(\mathcal{W}) \in \text{Rel}^v(\varphi \left[ \vec{\varphi}/\vec{\zeta} \right])$  is given by  $\text{fold}(\mathcal{W})^\varepsilon = \{(\text{fold}(W), v) \mid (W, v) \in \mathcal{W}^\varepsilon\}$  for  $\varepsilon \in \{+, -\}$ .

We recall the statement of Tarski's fix-point theorem.

**Theorem 18.** *Let  $S$  and  $T$  be complete meet semi-lattices and let  $f : S \times T \rightarrow T$  be a monotone function. For  $x \in S$ , let  $g(x)$  be the meet of the set  $\{y \in T \mid f(x, y) \leq y\}$ . Then the function  $g$  is monotone and satisfies  $f(x, g(x)) = g(x)$  for each  $x \in S$ .*

Applying this theorem we obtain, by induction on types, the following property.

**Lemma 19.** *For any type  $\sigma$  and any positive type  $\varphi$ , the maps  $\mathcal{R}(\sigma)_{\vec{\zeta}}$  and  $\mathcal{V}(\varphi)_{\vec{\zeta}}$  are monotone wrt. the  $\sqsubseteq$  order relation. If  $\psi$  is a positive type,  $\vec{\zeta} = (\zeta_1, \dots, \zeta_n, \zeta)$  is a repetition-free list of type variables containing all the free variables of  $\psi$  and  $\varphi = \mathbf{Fix} \zeta \cdot \psi$  and  $\vec{\mathcal{V}} = (\mathcal{V}_1, \dots, \mathcal{V}_n)$  is a list of pairs of relations such that  $\mathcal{V}_i \in \text{Rel}^v(\varphi_i)$  for each  $i$ , then  $\mathcal{V} = \mathcal{R}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}})$  satisfies  $\mathcal{V} = \text{fold}(\mathcal{R}(\psi)_{\vec{\zeta}, \zeta})(\vec{\mathcal{V}}, \mathcal{V})$ .*

**3.4.3. Some useful closeness lemmas.** We state and prove a series of lemmas expressing that our pairs of relations are closed under various syntactic and semantic operations.

**Lemma 20.** *Let  $M$  and  $M'$  be terms such that  $\vdash M : \mathbf{1}$  and  $\vdash M' : \mathbf{1}$ . If  $M \rightarrow_w M'$  then  $\text{Red}_{M, ()}^\infty = \text{Red}_{M', ()}^\infty$ .*

This is straightforward since any reduction path from  $M$  to  $()$  must start with the step  $M \rightarrow_w M'$ , and this is a probability 1 step.

**Lemma 21.** *Let  $\varphi$  be a closed positive type and let  $\sigma$  be a closed type. Let  $(M, u) \in \mathcal{R}(\varphi)^{-\varepsilon}$  and  $(R, r) \in \mathcal{R}(\varphi \multimap \sigma)^\varepsilon$ . Then  $(\langle R \rangle M, r(u)) \in \mathcal{R}(\sigma)^\varepsilon$ .*

*Proof.* We can write  $\sigma = \varphi_1 \multimap \dots \multimap \varphi_n \multimap \psi$  for some  $n$  and  $\varphi_1, \dots, \varphi_n, \psi$  positive and closed. Given  $(V_i, v_i) \in \mathcal{V}(\varphi_i)^{-\varepsilon}$  for  $i = 1, \dots, n$ , we have to prove that

$$(\langle R \rangle M V_1 \dots V_n, r(u)(v_1) \dots (v_n)) \in \overline{(\mathcal{V}(\psi))}^\varepsilon$$

so let  $(T, t) \in (\mathcal{V}(\psi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ , we have to prove that

$$(\langle T \rangle (\langle R \rangle M V_1 \dots V_n), t(r(u)(v_1) \dots (v_n))) \in \mathcal{R}(\mathbf{1}).$$

Let  $S = \lambda x^\varphi \langle T \rangle (\langle R \rangle x V_1 \dots V_n)$  so that  $\vdash S : \varphi \multimap \mathbf{1}$ . Similarly let  $s \in \mathbf{P}[\varphi \multimap \mathbf{1}]$  be the linear morphism defined by  $s(u') = t(r(u')(v_1) \dots (v_n))$  (the fact that  $s$  so defined is actually a morphism in  $\mathbf{Pcoh}$  results from the symmetric monoidal closeness of that category and from the fact that  $r$  and  $t$  are morphisms in  $\mathbf{Pcoh}$ ). Let  $(V, v) \in \mathcal{V}(\varphi)^{-\varepsilon}$ , we have  $(\langle R \rangle V, r(v)) \in \mathcal{R}(\sigma)^\varepsilon$  and hence  $(\langle R \rangle V V_1 \dots V_n, r(v)(v_1) \dots (v_n)) \in \mathcal{R}(\psi)^\varepsilon$ . Therefore

$$(\langle T \rangle (\langle R \rangle V V_1 \dots V_n), t(r(v)(v_1) \dots (v_n))) \in \mathcal{R}(\mathbf{1})$$

since we have assumed that  $(T, t) \in (\mathcal{V}(\psi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ . Since  $t(r(v)(v_1) \dots (v_n)) = s(v)$ , and by Lemma 20, it follows that  $(\langle S \rangle V, s(v)) \in \mathcal{R}(\mathbf{1})$ . Hence  $(S, s) \in \mathcal{R}(\varphi \multimap \mathbf{1})^\varepsilon$  and therefore  $(\langle S \rangle M, s(u)) \in \mathcal{R}(\mathbf{1})$  since we have  $(M, u) \in \mathcal{R}(\varphi)^{-\varepsilon}$ .

We finish the proof by observing that  $s(u) = t(r(u)(v_1) \dots (v_n))$  and by showing that

$$\text{Red}_{\langle T \rangle \langle R \rangle M V_1 \dots V_n, ()}^\infty = \text{Red}_{\langle S \rangle M, ()}^\infty$$

For this it suffices to observe (by inspection of the reduction rules) that each reduction path  $\pi$  from  $\langle T \rangle (\langle R \rangle M V_1 \dots V_n)$  to  $()$  is of shape  $\pi = \lambda \rho$  where



- $\lambda$  is a reduction path

$$\langle T \rangle (\langle R \rangle M_1 V_1 \cdots V_n) \xrightarrow{p_1} \langle T \rangle (\langle R \rangle M_2 V_1 \cdots V_n) \xrightarrow{p_2} \cdots \xrightarrow{p_k} \langle T \rangle (\langle R \rangle M_{k+1} V_1 \cdots V_n)$$

where  $M_1 = M$ ,  $M_{k+1}$  is a value  $V$  and  $M = M_1 \xrightarrow{p_1} M_2 \xrightarrow{p_2} \cdots \xrightarrow{p_k} M_{k+1} = V$

- and  $\rho$  is a reduction path from  $\langle T \rangle (\langle R \rangle V V_1 \cdots V_n)$  to  $()$ .

Then we have  $\langle S \rangle M = \langle S \rangle M_1 \xrightarrow{p_1} \langle S \rangle M_2 \xrightarrow{p_2} \cdots \xrightarrow{p_k} \langle S \rangle V \xrightarrow{1} \langle T \rangle (\langle R \rangle V V_1 \cdots V_n)$ , the last step resulting from the definition of  $S$ . In that way, we have defined a probability preserving bijection between the reduction paths from  $\langle T \rangle (\langle R \rangle M_1 V_1 \cdots V_n)$  to  $()$  and the reduction paths from  $\langle S \rangle M$  to  $()$ , proving our contention.  $\square$

**Lemma 22.** *Let  $\varphi_i$  be closed positive types and  $(M_i, u_i) \in \mathcal{R}(\varphi_i)^\varepsilon$  for  $i = 1, 2$ . Then  $((M_1, M_2), u_1 \otimes u_2) \in \mathcal{R}(\varphi_1 \otimes \varphi_2)^\varepsilon$ .*

*Proof.* Let  $(T, t) \in (\mathcal{V}(\varphi_1 \otimes \varphi_2) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ , we must prove that  $(\langle T \rangle (M_1, M_2), t(u_1 \otimes u_2)) \in \mathcal{R}(\mathbf{1})$ . Let  $S = \lambda x_1^{\varphi_1} \lambda x_2^{\varphi_2} \langle T \rangle (x_1, x_2)$  and  $s \in \mathbf{P}[\varphi_1 \multimap (\varphi_2 \multimap \mathbf{1})]$  be defined by  $s(u_1)(u_2) = t(u_1 \otimes u_2)$  (again,  $s$  is a morphism in  $\mathbf{Pcoh}$  by symmetric monoidal closeness of that category). It is clear that  $(S, s) \in (\mathcal{V}(\varphi_1) \multimap (\mathcal{V}(\varphi_2) \multimap \mathcal{R}(\mathbf{1})))^{-\varepsilon}$ . By Lemma 21 we get  $(\langle S \rangle M_1, s(u_1)) \in (\mathcal{V}(\varphi_2) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$  and then  $(\langle S \rangle M_1 M_2, t(u_1 \otimes u_2)) \in \mathcal{R}(\mathbf{1})$ . Observing that there is a probability preserving bijection between the reduction paths from  $\langle S \rangle M_1 M_2$  to  $()$  and the reduction paths from  $\langle T \rangle (M_1, M_2)$  to  $()$ , we conclude that  $(\langle T \rangle (M_1, M_2), t(u_1 \otimes u_2)) \in \mathcal{R}(\mathbf{1})$  as contended (in both terms one has to reduce first  $M_1$  and then  $M_2$  to a value).  $\square$

**Lemma 23.** *Let  $\varphi_1$  and  $\varphi_2$  be closed positive types. If  $(M, u) \in \mathcal{R}(\varphi_1 \otimes \varphi_2)^\varepsilon$  then  $(\text{pr}_i M, \text{pr}_i(u)) \in \mathcal{R}(\varphi_i)^\varepsilon$ .*

*Proof.* Let  $(T, t) \in (\mathcal{V}(\varphi_i) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ , we have to prove that  $(\langle T \rangle \text{pr}_i M, t(\text{pr}_i(u))) \in \mathcal{R}(\mathbf{1})$ . Let  $S = \lambda x^{\varphi_1 \otimes \varphi_2} \langle T \rangle \text{pr}_i x$  and  $s \in \mathbf{P}[\varphi_1 \otimes \varphi_2 \multimap \mathbf{1}]$  be defined by  $s(u_0) = t(\text{pr}_i(u_0))$  for all  $u_0 \in \mathbf{P}[\varphi_1 \otimes \varphi_2]$ . Let  $(W, w) \in \mathcal{V}(\varphi_1 \otimes \varphi_2)^\varepsilon$ , which means that  $W = (V_1, V_2)$  and  $w = v_1 \otimes v_2$  with  $(V_j, v_j) \in \mathcal{V}(\varphi_j)^\varepsilon$  for  $j = 1, 2$ . We have  $\langle S \rangle W \rightarrow_w \langle T \rangle V_i$  and  $s(w) = t(v_i)$  and we know that  $(\langle T \rangle V_i, t(v_i)) \in \mathcal{R}(\mathbf{1})$  from which it follows by Lemma 20 that  $(\langle S \rangle W, s(w)) \in \mathcal{R}(\mathbf{1})$ . So we have proven that  $(S, s) \in (\mathcal{V}(\varphi_1 \otimes \varphi_2) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$  and hence  $(\langle S \rangle M, s(u)) \in \mathcal{R}(\mathbf{1})$ . We have  $s(u) = t(\text{pr}_i(u))$ . Moreover we have a probability preserving bijection between the reduction paths from  $\langle T \rangle \text{pr}_i M$  to  $()$  and the reduction paths from  $\langle S \rangle M$  to  $()$ , and hence we have  $(\langle T \rangle \text{pr}_i M, t(\text{pr}_i(u))) \in \mathcal{R}(\mathbf{1})$  as contended.

Indeed, any reduction path  $\pi$  from  $\langle T \rangle \text{pr}_i M$  to  $()$  has shape  $\pi = \lambda \rho$  where  $\lambda$  is a reduction path  $\langle T \rangle \text{pr}_i M = \langle T \rangle \text{pr}_i M_1 \xrightarrow{p_1} \langle T \rangle \text{pr}_i M_2 \xrightarrow{p_2} \cdots \xrightarrow{p_k} \langle T \rangle \text{pr}_i W \xrightarrow{1} \langle T \rangle V_i$  (with  $W = (V_1, V_2)$ ) and  $\rho$  is a reduction path from  $\langle T \rangle V_i$  to  $()$ . The first steps  $\lambda$  of this reduction are determined by the reduction path  $M = M_1 \xrightarrow{p_1} \cdots \xrightarrow{p_k} W$  from  $M$  to the value  $W$ . This reduction path determines uniquely the reduction path  $\langle S \rangle M = \langle S \rangle M_1 \xrightarrow{p_1} \cdots \xrightarrow{p_k} \langle S \rangle W \xrightarrow{1} \langle T \rangle \text{pr}_i W \xrightarrow{1} \langle T \rangle V_i$  followed by the reduction  $\rho$  from  $\langle T \rangle V_i$  to  $()$  by  $\rho$ .  $\square$

**Lemma 24.** *Let  $\varphi_1$  and  $\varphi_2$  be closed positive types and let  $(M, u) \in \mathcal{R}(\varphi_i)^\varepsilon$  for  $i = 1$  or  $i = 2$ . Then  $(\text{in}_i M, \text{in}_i(u)) \in \mathcal{R}(\varphi_1 \oplus \varphi_2)^\varepsilon$ .*

*Proof.* Let  $(T, t) \in (\mathcal{V}(\varphi_1 \oplus \varphi_2) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ , we must prove that  $(\langle T \rangle \text{in}_i M, t(\text{in}_i(u))) \in \mathcal{R}(\mathbf{1})$ . Let  $S = \lambda x^{\varphi_i} \langle T \rangle \text{in}_i(x)$  and let  $s \in \mathbf{P}[\varphi_i \multimap \mathbf{1}]$ . It is clear that  $(S, s) \in (\mathcal{V}(\varphi_i) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$  and it follows that  $(\langle S \rangle M, s(u)) \in \mathcal{R}(\mathbf{1})$  which implies  $(\langle T \rangle \text{in}_i M, t(\text{in}_i(u))) \in \mathcal{R}(\mathbf{1})$  by the usual bijective and probability preserving bijection on reductions.  $\square$



**Lemma 25.** *Let  $\varphi_1$  and  $\varphi_2$  be closed positive type and  $\sigma$  be a closed type. Let  $(M, u) \in \mathcal{R}(\varphi_1 \oplus \varphi_2)^\varepsilon$ . For  $i = 1, 2$ , let  $R_i$  be a term such that  $y_i : \varphi_i \vdash R_i : \sigma$  and assume that  $(\lambda x_i^{\varphi_i} R_i, r_i) \in \mathcal{R}(\varphi_i \multimap \sigma)^{-\varepsilon}$ . Then  $(\text{case}(M, y_1 \cdot R_1, y_2 \cdot R_2), \text{case}(r_1, r_2)(u)) \in \mathcal{R}(\sigma)^{-\varepsilon}$ .*

*Proof.* We can write  $\sigma = \psi_1 \multimap \dots \multimap \psi_k \multimap \psi$  where  $\psi$  and the  $\psi_j$ 's are closed and positive types. Given  $(W_j, w_j) \in \mathcal{V}(\psi_j)^\varepsilon$ , we have to prove that

$$(\langle \text{case}(M, y_1 \cdot R_1, y_2 \cdot R_2) \rangle \vec{W}, \text{case}(r_1, r_2)(u)(\vec{w})) \in \mathcal{V}(\psi)^{-\varepsilon} \quad (3.2)$$

so let  $(T, t) \in (\mathcal{V}(\psi) \multimap \mathcal{R}(\mathbf{1}))^\varepsilon$ , our goal is to prove that

$$(\langle T \rangle \langle \text{case}(M, y_1 \cdot R_1, y_2 \cdot R_2) \rangle \vec{W}, t(\text{case}(r_1, r_2)(u)(\vec{w}))) \in \mathcal{R}(\mathbf{1}). \quad (3.3)$$

Let  $S = \lambda x^{\varphi_1 \oplus \varphi_2} \langle T \rangle \langle \text{case}(x, y_1 \cdot R_1, y_2 \cdot R_2) \rangle \vec{W}$  and  $s \in \mathcal{P}[\varphi_1 \oplus \varphi_2 \multimap \mathbf{1}]$  be defined by  $s(u_0) = t(\text{case}(r_1, r_2)(u_0)(\vec{w}))$  for each  $u_0 \in \mathcal{P}[\varphi_1 \oplus \varphi_2]$ . Then we have  $(S, s) \in (\mathcal{V}(\varphi_1 \oplus \varphi_2) \multimap \mathcal{R}(\mathbf{1}))^\varepsilon$ . Let indeed  $i \in \{1, 2\}$  and let  $(V, v) \in \mathcal{V}(\varphi_i)^{-\varepsilon}$  so that  $(\text{in}_i V, \text{in}_i(v)) \in \mathcal{V}(\varphi_1 \oplus \varphi_2)^{-\varepsilon}$ . We have  $\langle S \rangle \text{in}_i V \rightarrow_w^* \langle T \rangle \langle R_i[V/y_i] \rangle \vec{W}$  and  $s(\text{in}_i(v)) = t(r_i(v)(\vec{w}))$  and, by our assumptions and Lemma 20,  $(R_i[V/y_i], r_i(v)) \in \mathcal{R}(\sigma)^{-\varepsilon}$  and hence  $(\langle R_i[V/y_i] \rangle \vec{W}, r_i(v)(\vec{w})) \in \mathcal{V}(\psi)^{-\varepsilon}$ . By Lemma 20 it follows that  $(\langle S \rangle \text{in}_i V, s(\text{in}_i(v))) \in \mathcal{R}(\mathbf{1})$  and hence  $(S, s) \in (\mathcal{V}(\varphi_1 \oplus \varphi_2) \multimap \mathcal{R}(\mathbf{1}))^\varepsilon$  as contended.

Therefore  $(\langle S \rangle M, s(u)) \in \mathcal{R}(\mathbf{1})$ . There is a bijective and probability preserving correspondence between the reductions from  $\langle S \rangle M$  to  $()$  and the reductions from  $\langle T \rangle \langle \text{case}(M, x_1 \cdot \langle R_1 \rangle x_1, x_2 \cdot \langle R_2 \rangle x_2) \rangle \vec{W}$  to  $()$ : as usual such reductions start with a reduction  $M = M_1 \xrightarrow{p_1} M_2 \xrightarrow{p_2} \dots \xrightarrow{p_k} M_k = \text{in}_i V$  where  $i \in \{1, 2\}$  and  $V$  is a value of type  $\varphi_i$  and (after a few  $\rightarrow_w$ -steps) continue with a reduction from  $\langle T \rangle \langle R_i \rangle V \vec{W}$  to  $()$ . Therefore (3.3) holds and hence we have (3.2), this ends the proof of the lemma.  $\square$

**Lemma 26.** *Let  $\sigma$  be a closed type and let  $(M, u) \in \mathcal{R}(!\sigma)^\varepsilon$ . We have  $(\text{der}(M), \text{der}(u)) \in \mathcal{R}(\sigma)^\varepsilon$ .*

*Proof.* We can write  $\sigma = \psi_1 \multimap \dots \multimap \psi_k \multimap \psi$  where  $\psi$  and the  $\psi_j$ 's are closed and positive types. Given  $(W_j, w_j) \in \mathcal{V}(\psi_j)^{-\varepsilon}$ , we have to prove that

$$(\langle \text{der}(M) \rangle \vec{W}, \text{der}(u)(\vec{w})) \in \mathcal{V}(\psi)^\varepsilon \quad (3.4)$$

so let  $(T, t) \in (\mathcal{V}(\psi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ , our goal is to prove that

$$(\langle T \rangle \langle \text{der}(M) \rangle \vec{W}, t(\text{der}(u)(\vec{w}))) \in \mathcal{R}(\mathbf{1}). \quad (3.5)$$

We set  $S = \lambda x^{!\sigma} \langle T \rangle \langle \text{der}(x) \rangle \vec{W}$  and we define  $s \in \mathcal{P}![\sigma \multimap \mathbf{1}]$  by  $s(u_0) = t(\text{der}(u_0)(\vec{w}))$  for all  $u_0 \in \mathcal{P}![\sigma]$ , and we prove that  $(S, s) \in (\mathcal{V}(!\sigma) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$  as in the proof of Lemme 25 (for instance). We finish the proof in the same way, showing (3.5) by establishing a bijective and probability preserving correspondence between reductions. Our contention (3.4) follows.  $\square$

**Lemma 27.** *Let  $\varphi$  be a closed positive type of shape  $\varphi = \text{Fix } \zeta \cdot \psi$ . If  $(M, u) \in \mathcal{R}(\varphi)^\varepsilon$  then  $(\text{unfold}(M), u) \in \mathcal{R}(\psi[\varphi/\zeta])^\varepsilon$ .*

*Proof.* Let  $(T, t) \in (\mathcal{V}(\psi[\varphi/\zeta]) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ , we must prove that  $(\langle T \rangle \text{unfold}(M), u) \in \mathcal{R}(\mathbf{1})$ . As usual one defines  $S = \lambda x^\varphi \langle T \rangle \text{unfold}(x)$  and one proves that  $(S, t) \in (\mathcal{V}(\varphi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ . This results from the fact that if  $(V, v) \in \mathcal{V}(\varphi)^\varepsilon$  then  $V = \text{fold}(W)$  with  $(W, v) \in \mathcal{V}(\psi[\varphi/\zeta])^\varepsilon$ , from Lemma 20 and from the fact that  $\text{unfold}(\text{fold}(W)) \rightarrow_w W$  (and of course from our assumption on  $(T, t)$ ). It follows that  $(\langle S \rangle M, t(u)) \in \mathcal{R}(\mathbf{1})$  from which we deduce  $(\langle T \rangle \text{unfold}(M), u) \in \mathcal{R}(\mathbf{1})$  by the usual reasoning involving a bijective probability preserving correspondence on reductions.  $\square$

**Lemma 28.** *Let  $\varphi$  be a closed positive type of shape  $\varphi = \mathbf{Fix} \zeta \cdot \psi$ . If  $(M, u) \in \mathcal{R}(\psi[\varphi/\zeta])^\varepsilon$  then  $(\text{fold}(M), u) \in \mathcal{R}(\varphi)^\varepsilon$ .*

*Proof.* Let  $(T, t) \in (\mathcal{V}(\varphi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ , we must prove that  $(\langle T \rangle \text{fold}(M), u) \in \mathcal{R}(\mathbf{1})$ . As usual one defines  $S = \lambda x^{\psi[\varphi/\zeta]} \langle T \rangle \text{fold}(x)$  and one proves that  $(S, t) \in (\mathcal{V}(\psi[\varphi/\zeta]) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ . This results easily from the fact that, if  $(V, v) \in \mathcal{V}(\psi[\varphi/\zeta])^\varepsilon$  then  $(\text{fold}(V), v) \in \mathcal{V}(\varphi)^\varepsilon$ , from Lemma 20 and from our assumption about  $(T, t)$ . Therefore we have  $(\langle S \rangle M, t(u)) \in \mathcal{R}(\mathbf{1})$  from which we deduce  $(\langle T \rangle \text{fold}(M), u) \in \mathcal{R}(\mathbf{1})$  by the usual reasoning.  $\square$

**Lemma 29.** *Let  $\sigma$  be a closed type and let  $M$  be a closed term of type  $\sigma$ . Then  $(M, 0) \in \mathcal{R}(\sigma)^\varepsilon$  and, if  $D \subseteq \mathbf{P}[\sigma]$  is directed and satisfies  $\forall u \in D \ (M, u) \in \mathcal{R}(\sigma)^\varepsilon$  then  $(M, \sup D) \in \mathcal{R}(\sigma)^\varepsilon$ . Last, if  $(M, u) \in \mathcal{R}(\sigma)^\varepsilon$  and  $u' \leq u$  then  $(M, u') \in \mathcal{R}(\sigma)^\varepsilon$ .*

*Proof.* We can write  $\sigma = \varphi_1 \multimap \dots \multimap \varphi_n \multimap \psi$  for some  $n$  and  $\varphi_1, \dots, \varphi_n, \psi$  positive and closed. Let us prove the second statement. For  $i = 1, \dots, n$ , let  $(V_i, v_i) \in \mathcal{V}(\varphi_i)^{-\varepsilon}$ , we must prove that  $(\langle M \rangle V_1 \dots V_n, (\sup D)(v_1) \dots (v_n)) \in \overline{\mathcal{V}(\psi)}^\varepsilon$ , knowing that

$$\forall u \in D \quad (\langle M \rangle V_1 \dots V_n, u(v_1) \dots (v_n)) \in \overline{\mathcal{V}(\psi)}^\varepsilon.$$

This results from the fact that, given  $t \in \mathbf{P}[\psi \multimap \mathbf{1}]$ , the map  $u \mapsto t(u(v_1) \dots (v_n))$  is Scott continuous from  $\mathbf{P}[\varphi]$  to  $[0, 1]$ . The first statement of the lemma results from the fact that this function maps 0 to 0. The last one results from the fact that this function is monotone.  $\square$

**3.4.4. Uniqueness of the relation.** With any closed type  $\sigma$  we have associated a pair of relations  $\mathcal{R}(\sigma)$ . We need now to prove that this pair satisfies  $\mathcal{R}(\sigma)^+ = \mathcal{R}(\sigma)^-$  so that we have actually associated a unique relation with any type.

To this end we prove first that  $\mathcal{R}(\sigma)^+ \subseteq \mathcal{R}(\sigma)^-$ . Defining, for any pair of relations  $\mathcal{R}$ , the relation  $\mathcal{R}^{\text{op}}$  as  $(\mathcal{R}^+, \mathcal{R}^-)$ , this amounts to proving that  $\mathcal{R}(\sigma) \subseteq \mathcal{R}(\sigma)^{\text{op}}$ . We use the same notation for the elements of  $\text{Rel}^{\text{v}}(\varphi)$  for  $\varphi$  positive.

For the next lemma, we use the same notational conventions as above.

**Lemma 30.** *Let  $\vec{\mathcal{V}}$  be a list of pairs of relations such that  $\mathcal{V}_i \in \text{Rel}^{\text{v}}(\varphi_i)$  and  $\mathcal{V}_i \subseteq \mathcal{V}_i^{\text{op}}$  for each  $i$ . Then  $\mathcal{R}(\sigma)_{\vec{\zeta}}(\vec{\mathcal{V}}) \subseteq \mathcal{R}(\sigma)_{\vec{\zeta}}(\vec{\mathcal{V}})^{\text{op}}$  and  $\mathcal{V}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}}) \subseteq \mathcal{V}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}})^{\text{op}}$ .*

*Proof.* The proof is by induction on types. All cases result straightforwardly from the monotony of the logical operations on pairs of relations, but the case of fixpoints of types. So assume that  $\varphi = \mathbf{Fix} \zeta \cdot \psi$ , let  $\mathcal{V} = \mathcal{V}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}})$  and let us prove that  $\mathcal{V} \subseteq \mathcal{V}^{\text{op}}$ . For this, because of the definition of  $\mathcal{V}$  as a glb (3.1), it suffices to show that

$$\text{fold}(\mathcal{V}(\psi)_{\vec{\zeta}, \zeta}(\vec{\mathcal{V}}, \mathcal{V}^{\text{op}})) \subseteq \mathcal{V}^{\text{op}}.$$

By Lemma 19 and our assumption on the  $\mathcal{V}_i$ 's we have

$$\text{fold}(\mathcal{V}(\psi)_{\vec{\zeta}, \zeta}(\vec{\mathcal{V}}, \mathcal{V}^{\text{op}})) \subseteq \text{fold}(\mathcal{V}(\psi)_{\vec{\zeta}, \zeta}(\vec{\mathcal{V}}^{\text{op}}, \mathcal{V}^{\text{op}})).$$

By inductive hypothesis we have  $\text{fold}(\mathcal{V}(\psi)_{\vec{\zeta}, \zeta}(\vec{\mathcal{V}}^{\text{op}}, \mathcal{V}^{\text{op}})) \subseteq \text{fold}(\mathcal{V}(\psi)_{\vec{\zeta}, \zeta}(\vec{\mathcal{V}}, \mathcal{V}))^{\text{op}} = \mathcal{V}^{\text{op}}$  since  $\mathcal{V} = \text{fold}(\mathcal{V}(\psi)_{\vec{\zeta}, \zeta}(\vec{\mathcal{V}}, \mathcal{V}))$  by Lemma 19.  $\square$

We are left with proving the converse property, namely that  $\mathcal{R}(\sigma)^{\text{op}} \sqsubseteq \mathcal{R}(\sigma)$  for each closed type  $\sigma$ . This requires a bit more work, and is based on a notion of “finite” approximation of elements of the model, that we define by syntactic means as follows.

**3.4.5. Restriction operators.** By lexicographic ordering on pairs<sup>7</sup>  $(n, \sigma)$  where  $n \in \mathbb{N}$  and  $\sigma$  is a type, we define closed terms  $\mathbf{p}(n, \sigma)$  and  $\mathbf{p}^\vee(n, \varphi)$  (when  $\varphi$  is positive) typed as follows:  $\vdash \mathbf{p}(n, \sigma) : !\sigma \multimap \sigma$  and  $\vdash \mathbf{p}^\vee(n, \varphi) : \varphi \multimap \varphi$ .

$$\begin{aligned} \mathbf{p}(n, \varphi) &= \lambda x.^! \varphi \langle \mathbf{p}^\vee(n, \varphi) \rangle \text{der}(x) \\ \mathbf{p}(n, \varphi \multimap \sigma) &= \lambda f.^! (\varphi \multimap \sigma) \lambda x.^! \varphi \langle \mathbf{p}(n, \sigma) \rangle (\langle \text{der}(f) \rangle \langle \mathbf{p}^\vee(n, \varphi) \rangle x) \\ \mathbf{p}^\vee(n, \mathbf{1}) &= \lambda x.^! x \\ \mathbf{p}^\vee(n, !\sigma) &= \lambda x.^! \sigma (\langle \mathbf{p}(n, \sigma) \rangle x) \\ \mathbf{p}^\vee(n, \varphi_1 \otimes \varphi_2) &= \lambda x.^! \varphi_1 \otimes \varphi_2 (\langle \mathbf{p}^\vee(n, \varphi_1) \rangle \text{pr}_1 x, \langle \mathbf{p}^\vee(n, \varphi_2) \rangle \text{pr}_2 x) \\ \mathbf{p}^\vee(n, \varphi_1 \oplus \varphi_2) &= \lambda x.^! \varphi_1 \oplus \varphi_2 \text{case}(x, x_1 \cdot \text{in}_1 \langle \mathbf{p}^\vee(n, \varphi_1) \rangle x_1, x_2 \cdot \text{in}_2 \langle \mathbf{p}^\vee(n, \varphi_2) \rangle x_2) \\ \mathbf{p}^\vee(0, \mathbf{Fix} \zeta \cdot \varphi) &= \lambda x.^! \text{Fix} \zeta \cdot \varphi \Omega^{\text{Fix} \zeta \cdot \varphi} \\ \mathbf{p}^\vee(n+1, \mathbf{Fix} \zeta \cdot \varphi) &= \lambda x.^! \text{Fix} \zeta \cdot \varphi \text{fold}(\langle \mathbf{p}^\vee(n, \varphi [\mathbf{Fix} \zeta \cdot \varphi / \zeta]) \rangle \text{unfold}(x)) \end{aligned}$$

We describe now the interpretation of these terms: we give an explicit description of the matrices  $[\mathbf{p}(n, \sigma)]$  and  $[\mathbf{p}^\vee(n, \varphi)]$ . To this end, we define two families of sets  $\mathbf{I}(n, \sigma) \subseteq [|\sigma|]$  and  $\mathbf{I}^\vee(n, \varphi) \subseteq [|\varphi|]$  by induction on  $(n, \sigma)$  and  $(n, \varphi)$  (where  $n \in \mathbb{N}$ ,  $\varphi$  is a closed positive type and  $\sigma$  is a closed type).

- $\mathbf{I}^\vee(n, !\sigma) = \mathcal{M}_{\text{fin}}(\mathbf{I}(n, \sigma))$
- $\mathbf{I}^\vee(n, \varphi_1 \otimes \varphi_2) = \mathbf{I}^\vee(n, \varphi_1) \times \mathbf{I}^\vee(n, \varphi_2)$
- $\mathbf{I}^\vee(n, \varphi_1 \oplus \varphi_2) = \{1\} \times \mathbf{I}^\vee(n, \varphi_1) \cup \{2\} \times \mathbf{I}^\vee(n, \varphi_2)$
- $\mathbf{I}^\vee(0, \mathbf{Fix} \zeta \cdot \psi) = \emptyset$
- $\mathbf{I}^\vee(n+1, \mathbf{Fix} \zeta \cdot \psi) = \mathbf{I}^\vee(n, \varphi [\mathbf{Fix} \zeta \cdot \psi / \zeta])$
- $\mathbf{I}(n, \varphi) = \mathbf{I}^\vee(n, \varphi)$
- $\mathbf{I}(n, \varphi \multimap \sigma) = \mathbf{I}^\vee(n, \varphi) \times \mathbf{I}(n, \sigma)$ .

**Lemma 31.** *Let  $n \in \mathbb{N}$ ,  $\varphi$  be a closed positive type and  $\sigma$  be a closed type. One has*

$$[\mathbf{p}(n, \varphi)]_{(a,b)} = \begin{cases} 1 & \text{if } a = b \in \mathbf{I}^\vee(n, \varphi) \\ 0 & \text{otherwise.} \end{cases} \quad [\mathbf{p}(n, \sigma)]_{(c,b)} = \begin{cases} 1 & \text{if } c = [b] \text{ and } b \in \mathbf{I}(n, \sigma) \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By Theorem 16, for a closed positive type  $\varphi$  and for  $u \in \mathbf{P}^![\varphi]^!$ , it suffices to prove that

$$[\mathbf{p}^\vee(n, \varphi)](u)_a = \begin{cases} u_a & \text{if } a \in \mathbf{I}^\vee(n, \varphi) \\ 0 & \text{otherwise} \end{cases}$$

And for a closed type  $\sigma$  and for  $u \in \mathbf{P}[\sigma]$ , it suffices to prove

$$[\mathbf{p}(n, \sigma)](u^!)_a = \begin{cases} u_a & \text{if } a \in \mathbf{I}(n, \sigma) \\ 0 & \text{otherwise} \end{cases}$$

Both statements are easily proved by induction. □

<sup>7</sup>This definition as well as our reasoning below features some similarities with *step-indexing* that we would like to understand better.

We need now to prove that, given  $u \in \mathbf{P}[\sigma]$ , the sequence  $[\mathbf{p}(n, \sigma)](u^!)$  is monotone and has  $u$  as lub.

**Lemma 32.** *For a closed type  $\sigma$  and a closed positive type  $\varphi$ , we have*

$$\forall n \in \mathbb{N} \quad \mathbf{I}(n, \sigma) \subseteq \mathbf{I}(n+1, \sigma) \text{ and } \mathbf{I}^v(n, \varphi) \subseteq \mathbf{I}^v(n+1, \varphi)$$

$$\bigcup_{n=0}^{\infty} \mathbf{I}(n, \sigma) = \llbracket \sigma \rrbracket \text{ and } \bigcup_{n=0}^{\infty} \mathbf{I}^v(n, \varphi) = \llbracket \varphi \rrbracket$$

*Proof.* The first statement is straightforward, by induction on  $(n, \sigma)$  and  $(n, \varphi)$ . For the second statement, we only have to prove the right-to-left inclusions. We define the *height*  $\mathbf{h}(a)$  of an element  $a$  of  $\llbracket \sigma \rrbracket$  or  $\llbracket \varphi \rrbracket$  as follows.

- $\mathbf{h}(*) = 1$
- $\mathbf{h}(a_1, a_2) = 1 + \max(\mathbf{h}(a_1), \mathbf{h}(a_2))$  (this definition is used when  $\varphi$  is a tensor and when  $\sigma$  is a linear implication)
- $\mathbf{h}(i, a) = 1 + \mathbf{h}(a)$
- $\mathbf{h}([a_1, \dots, a_k]) = 1 + \max(\mathbf{h}(a_1), \dots, \mathbf{h}(a_k))$

Then by induction on  $\mathbf{h}(a)$  one proves that

$$\forall a \in \llbracket \sigma \rrbracket \exists n \in \mathbb{N} \ a \in \mathbf{I}(n, \sigma)$$

and a similar statement for  $\varphi$ . We deal only with the statement relative to  $\mathbf{I}^v(n, \varphi)$ . The closed positive type  $\varphi$  is of shape

$$\varphi = \mathbf{Fix} \zeta_1 \cdots \mathbf{Fix} \zeta_k \cdot \psi$$

where  $\psi$  is not of shape  $\mathbf{Fix} \zeta \cdot \rho$ . We introduce auxiliary closed types  $\varphi_1, \dots, \varphi_k$  as follows:

$$\begin{aligned} \varphi_1 &= \varphi = \mathbf{Fix} \zeta_1 \cdots \mathbf{Fix} \zeta_k \cdot \psi \\ \varphi_2 &= \mathbf{Fix} \zeta_2 \cdots \mathbf{Fix} \zeta_k \cdot \psi [\varphi_1 / \zeta_1] \\ &\vdots \\ \varphi_{k+1} &= \psi [\varphi_1 / \zeta_1, \varphi_2 / \zeta_2, \dots, \varphi_k / \zeta_k] \end{aligned}$$

and all these types have the same interpretation in  $\mathbf{Pcoh}^!$ . The type  $\psi$  cannot be one of the type variables  $\zeta_i$  as otherwise we would have  $\llbracket \varphi \rrbracket = \emptyset$ , contradicting our assumption that  $a$  belongs to this set. Assume that  $\psi = !\sigma$  so that we must have  $a = [b_1, \dots, b_l]$  with  $b_i \in \llbracket \sigma' \rrbracket$  (where  $\sigma' = \sigma [\varphi_1 / \zeta_1, \varphi_2 / \zeta_2, \dots, \varphi_k / \zeta_k]$ ) for each  $i = 1, \dots, l$ . We have  $\mathbf{h}(b_i) < \mathbf{h}(a)$  for each  $i$  so that we can apply the inductive hypothesis: for each  $i$  there is  $n_i$  such that  $b_i \in \mathbf{I}(n_i, \sigma')$ . Using the monotonicity property (first statement of the lemma) and setting  $n = \max(n_1, \dots, n_l)$  we have  $b_i \in \mathbf{I}(n, \sigma')$  and hence  $a \in \mathbf{I}^v(n, !\sigma')$ . Therefore  $a \in \mathbf{I}^v(n+k, \varphi)$  (coming back to the definition of this set). The other cases are dealt with similarly.  $\square$

**Lemma 33.** *Let  $\sigma$  be a closed type and let  $\varphi$  be a closed positive type. If  $u \in \mathbf{P}[\sigma]$  then the sequence  $([\mathbf{p}(n, \sigma)](u))_{n \in \mathbb{N}}$  is monotone (in  $\mathbf{P}[\sigma]$ ) and has  $u$  as lub. If  $u \in \mathbf{P}[\varphi]$  then the sequence  $([\mathbf{p}^v(n, \varphi)](u))_{n \in \mathbb{N}}$  is monotone and has  $u$  as lub.*

*Proof.* Immediate consequence of Lemmas 31 and 32.  $\square$

3.4.6. *Main Inclusion Lemma.* Now we are in position of proving the key lemma in the proof of the uniqueness of relations.

**Lemma 34.** *Let  $\sigma$  be a closed type and let  $n \in \mathbb{N}$ . If  $(M, u) \in \mathcal{R}(\sigma)^-$  then  $(M, [\mathbf{p}(n, \sigma)](u^!)) \in \mathcal{R}(\sigma)^+$ . Let  $\varphi$  be a closed positive type and let  $n \in \mathbb{N}$ . If  $(V, v) \in \mathcal{V}(\varphi)^-$  then  $(V, [\mathbf{p}^\vee(n, \varphi)](v)) \in (\overline{\mathcal{V}(\varphi)})^+ = \mathcal{R}(\varphi)^+$ .*

*Proof.* By simultaneous lexicographic induction on  $(n, \sigma)$  and  $(n, \varphi)$ . The only case where “ $n$  decreases” in this induction is when  $\varphi = \mathbf{Fix} \zeta \cdot \psi$ , we start with this case.

Assume that  $\varphi = \mathbf{Fix} \zeta \cdot \psi$  and that  $(V, v) \in \mathcal{V}(\varphi)^-$ . If  $n = 0$  we have  $[\mathbf{p}^\vee(n, \varphi)](v) = 0$  and hence  $(V, [\mathbf{p}^\vee(n, \varphi)](v)) \in \mathcal{R}(\varphi)^+$  by Lemma 29. Assume that the implication holds for  $n$  and let us prove it for  $n + 1$ . Let  $(V, v) \in \mathcal{V}(\varphi)^-$ , that is  $V = \mathbf{fold}(W)$  with  $(W, v) \in \mathcal{V}(\psi[\varphi/\zeta])^-$ . We have  $[\mathbf{p}^\vee(n + 1, \mathbf{Fix} \zeta \cdot \psi)](v) = [\mathbf{p}^\vee(n, \psi[\varphi/\zeta])](v)$  by definition. By inductive hypothesis we have

$$(W, [\mathbf{p}^\vee(n, \psi[\varphi/\zeta])](v)) \in \mathcal{R}(\psi[\varphi/\zeta])^+ \quad (3.6)$$

and we must prove that  $(\mathbf{fold}(W), [\mathbf{p}^\vee(n, \psi[\varphi/\zeta])](v)) \in \mathcal{R}(\varphi)^+$ . Let  $(T, t) \in (\mathcal{V}(\varphi) \multimap \mathcal{R}(\mathbf{1}))^-$ , we must prove that  $(\langle T \rangle \mathbf{fold}(W), t([\mathbf{p}^\vee(n, \psi[\varphi/\zeta])](v))) \in \mathcal{R}(\mathbf{1})$ . Let  $S = \lambda x^{\psi[\varphi/\zeta]} \langle T \rangle \mathbf{fold}(x)$ , we have  $(S, t) \in (\mathcal{V}(\psi[\varphi/\zeta]) \multimap \mathcal{R}(\mathbf{1}))^-$  by Lemma 20 and therefore

$$(\langle S \rangle W, t([\mathbf{p}^\vee(n, \psi[\varphi/\zeta])](v))) \in \mathcal{R}(\mathbf{1})$$

by (3.6) and Lemma 21 and this implies  $(\langle T \rangle \mathbf{fold}(W), t([\mathbf{p}^\vee(n, \psi[\varphi/\zeta])](v))) \in \mathcal{R}(\mathbf{1})$  by Lemma 20.

Assume that  $\varphi = !\sigma$  and that  $(V, v) \in \mathcal{V}(!\sigma)^-$ , that is  $V = M^!$  and  $v = u^!$  with  $(M, u) \in \mathcal{R}(\sigma)^-$ . By inductive hypothesis we have  $(M, [\mathbf{p}(n, \sigma)](u^!)) \in \mathcal{R}(\sigma)^+$  and hence  $(!M, ([\mathbf{p}(n, \sigma)](u^!))^!) \in \mathcal{V}(!\sigma)^+$  and since  $([\mathbf{p}(n, \sigma)](u^!))^! = [\mathbf{p}^\vee(n, !\sigma)](u^!)$  we get

$$(V, [\mathbf{p}^\vee(n, !\sigma)](v)) \in \mathcal{V}(!\sigma)^+ \subseteq \mathcal{R}(!\sigma)^+$$

as expected.

Assume that  $\varphi = \varphi_1 \otimes \varphi_2$  and that  $(V, v) \in \mathcal{V}(\varphi_1 \otimes \varphi_2)^-$ , that is  $V = (V_1, V_2)$  and  $v = v_1 \otimes v_2$  with  $(V_i, v_i) \in \mathcal{V}(\varphi_i)^-$  for  $i = 1, 2$ . By inductive hypothesis we have  $(V_i, [\mathbf{p}(n, \varphi_i)](v_i)) \in \mathcal{R}(\varphi_i)^+$ . By Lemma 22 we get  $((V_1, V_2), [\mathbf{p}(n, \varphi_1)](v_1) \otimes [\mathbf{p}(n, \varphi_2)](v_2)) \in \mathcal{R}(\varphi_1 \otimes \varphi_2)^+$ , that is  $((V_1, V_2), [\mathbf{p}(n, \varphi_1 \otimes \varphi_2)](v_1 \otimes v_2)) \in \mathcal{R}(\varphi_1 \otimes \varphi_2)^+$ .

Assume that  $\varphi = \varphi_1 \oplus \varphi_2$  and  $(V, v) \in \mathcal{V}(\varphi_1 \oplus \varphi_2)^-$ . This means that for some  $i \in \{1, 2\}$ , one has  $V = \mathbf{in}_i W$  and  $v = \mathbf{in}_i(w)$  for  $(W, w) \in \mathcal{V}(\varphi_i)^-$ . By inductive hypothesis we have  $(W, [\mathbf{p}(n, \varphi_i)](w)) \in \mathcal{R}(\varphi_i)^+$  and hence  $(\mathbf{in}_i W, \mathbf{in}_i([\mathbf{p}(n, \varphi_i)](w))) \in \mathcal{R}(\varphi_1 \oplus \varphi_2)^+$  by Lemma 24, that is  $(\mathbf{in}_i W, [\mathbf{p}(n, \varphi_1 \oplus \varphi_2)](w)) \in \mathcal{R}(\varphi_1 \oplus \varphi_2)^+$ .

Assume that  $\sigma$  is a closed positive type  $\varphi$  and let  $(M, u) \in \mathcal{R}(\sigma)^-$ , we must prove that  $(M, [\mathbf{p}(n, \sigma)](u^!)) \in \mathcal{R}(\sigma)^+$  which follows directly from the definition of  $\mathbf{p}(n, \varphi)$  and from the inductive hypothesis.

Assume last that  $\sigma = \varphi \multimap \tau$  and that  $(M, u) \in \mathcal{R}(\varphi \multimap \tau)^-$ , we must prove that  $(M, [\mathbf{p}(n, \varphi \multimap \tau)](u^!)) \in \mathcal{R}(\varphi \Rightarrow \tau)^+$ . Let  $(V, v) \in \mathcal{V}(\varphi)^-$ , we must prove that

$$(\langle M \rangle V, [\mathbf{p}(n, \varphi \multimap \tau)](u^!)(v)) \in \mathcal{R}(\tau)^+ \quad (3.7)$$

which follows from the fact that  $[\mathbf{p}(n, \varphi \multimap \tau)](u^!)(v) = [\mathbf{p}(n, \tau)](u([\mathbf{p}^\vee(n, \varphi)](v)))^!$ . Indeed the inductive hypothesis applied to  $(n, \varphi)$  yields  $(V, [\mathbf{p}^\vee(n, \varphi)](v)) \in \mathcal{R}(\varphi)^+$  and hence  $(\langle M \rangle V, u([\mathbf{p}^\vee(n, \varphi)](v))) \in \mathcal{R}(\tau)^+$  by Lemma 21, from which we derive (3.7) by Lemma 33 and Lemma 29.  $\square$

**Lemma 35.** *For any closed type  $\sigma$  one has  $\mathcal{R}(\sigma)^- = \mathcal{R}(\sigma)^+$ .*

*Proof.* Immediate consequence of lemmas 29, 33 and 34.  $\square$

From now on we simply use the notation  $\mathcal{R}(\sigma)$  instead of  $\mathcal{R}(\sigma)^-$  and  $\mathcal{R}(\sigma)^+$ .

3.4.7. *Logical relation lemma.* We can prove now the main result of this section.

**Theorem 36** (Logical Relation Lemma). *Assume that  $x_1 : \varphi_1, \dots, x_k : \varphi_k \vdash M : \sigma$  and let  $(V_i, v_i) \in \mathcal{R}(\varphi_i)$  (where  $V_i$  is a value and  $v_i \in \mathbf{P}^1[\varphi_i]$ ) for  $i = 1, \dots, k$ . Then  $(M[V_1/x_1, \dots, V_k/x_k], [M]^{x_1, \dots, x_k}(v_1, \dots, v_k)) \in \mathcal{R}(\sigma)$ .*

*Remark.* One would expect to have rather assumptions of the shape “ $(V_i, v_i) \in \mathcal{V}(\varphi_i)$ ”; the problem is that we don’t know whether  $\mathcal{V}(\varphi_i)^+ = \mathcal{V}(\varphi_i)^-$ .

*Proof.* By induction on the typing derivation of  $M$ , that is, on  $M$ . We set  $\mathcal{P} = (x_1 : \varphi_1, \dots, x_k : \varphi_k)$  and, given a term  $R$ , we use  $R'$  for the term  $R[V_1/x_1, \dots, V_k/x_k]$ . We also use  $\vec{v}$  for the sequence  $v_1, \dots, v_k$  and  $\vec{x}$  for the sequence  $x_1, \dots, x_k$ .

The case  $M = x_i$  is straightforward.

Assume that  $M = N^!$  and that  $\varphi = !\sigma$  with  $\mathcal{P} \vdash N : \sigma$ . By inductive hypothesis we have  $(N', [N]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\sigma)$ . Therefore  $((N')^!, [N]^{\vec{x}}(\vec{v})^!) \in \mathcal{V}(!\sigma)^\varepsilon$  (for  $\varepsilon = +$  or  $\varepsilon = -$ )<sup>8</sup>. We have  $\mathcal{V}(!\sigma)^\varepsilon \subseteq \overline{\mathcal{V}(!\sigma)^\varepsilon} = \mathcal{R}(!\sigma)^\varepsilon = \mathcal{R}(!\sigma)$  and hence  $(M', [M]^{\vec{x}}(\vec{v})) \in \mathcal{R}(!\sigma)$  as contended.

Assume that  $M = (N_1, N_2)$  and  $\sigma = \psi_1 \otimes \psi_2$  with  $\mathcal{P} \vdash N_i : \psi_i$  for  $i = 1, 2$ . By inductive hypothesis we have  $(N'_i, [N_i]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\psi_i)$ . By Lemma 22 we get

$$((N'_1, N'_2), [(N_1, N_2)]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\psi_1 \otimes \psi_2)$$

as contended, since  $[(N_1, N_2)]^{\vec{x}}(\vec{v}) = [N_1]^{\vec{x}}(\vec{v}) \otimes [N_2]^{\vec{x}}(\vec{v})$ .

The case  $M = \text{in}_i N$  (and  $\sigma = \psi_1 \oplus \psi_2$ ) is handled similarly, using Lemma 24.

Assume that  $M = \text{fold}(N)$  and  $\sigma = \varphi = \text{Fix } \zeta \cdot \psi$  with  $\mathcal{P} \vdash N : \psi[\varphi/\zeta]$ . By inductive hypothesis we have  $(N', [N]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\psi[\varphi/\zeta])$  which implies  $(\text{fold}(N'), [N]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\varphi)$  by Lemma 28.

Assume that  $M = \lambda x^\varphi N$  and  $\sigma = \varphi \multimap \tau$ , with  $\mathcal{P}, x : \varphi \vdash N : \tau$ . We must prove that  $(\lambda x^\varphi N', [\lambda x^\varphi N]^{\vec{x}}(\vec{v})) \in (\mathcal{V}(\varphi) \multimap \mathcal{R}(\tau))^\varepsilon$  for an arbitrary  $\varepsilon \in \{-, +\}$ . So let  $(V, v) \in \mathcal{V}(\varphi)^{-\varepsilon}$ . Since  $\mathcal{V}(\varphi)^{-\varepsilon} \subseteq \mathcal{R}(\varphi)$ , we have  $(N'[V/x], [N]^{\vec{x}, x}(\vec{v}, v)) \in \mathcal{R}(\tau)$  by inductive hypothesis. It follows that  $(\langle \lambda x^\varphi N' \rangle V, [\lambda x^\varphi N]^{\vec{x}}(\vec{v})(v)) \in \mathcal{R}(\tau)$  by Lemma 20, proving our contention.

Assume that  $M = \langle R \rangle N$  with  $\mathcal{P} \vdash R : \varphi \multimap \sigma$  and  $\mathcal{P} \vdash N : \varphi$  where  $\varphi$  is a closed positive type. By inductive hypothesis we have  $(R', [R]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\varphi \multimap \sigma)$  and  $(N', [N]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\varphi)$  and hence  $(\langle R' \rangle N', [R]^{\vec{x}}(\vec{v})([N]^{\vec{x}}(\vec{v}))) \in \mathcal{R}(\sigma)$  by Lemma 21, that is  $(M', [M]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\sigma)$ .

Assume that  $M = \text{fix } x^{! \sigma} N$  with  $\mathcal{P}, x : !\sigma \vdash N : \sigma$ . The function  $f : \mathbf{P}[\sigma] \rightarrow \mathbf{P}[\sigma]$  defined by

$$f(u) = [N]^{\vec{x}, x}(\vec{v}, u^!)$$

<sup>8</sup>It is not clear whether  $\mathcal{V}(\varphi)^- = \mathcal{V}(\varphi)^+$  for any closed positive type  $\varphi$ , but we don’t need this property in this proof, so we leave this technical question unanswered.

is Scott continuous and we have  $[M]^{\vec{x}}(\vec{v}) = \sup_{k \in \mathbb{N}} f^k(0)$ . By induction on  $k$ , we prove that

$$\forall k \in \mathbb{N} \quad (M', f^k(0)) \in \mathcal{R}(\sigma). \quad (3.8)$$

The base case is proven by Lemma 29. Assume that  $(M', f^k(0)) \in \mathcal{R}(\sigma)$ . Choosing an arbitrary  $\varepsilon$ , we have  $((M')^!, (f^k(0))^!) \in \mathcal{V}(!\sigma)^\varepsilon \subseteq \mathcal{R}(!\sigma)$  and hence by our “outermost” inductive hypothesis we have  $(N'[(M')^!/x], f^{k+1}(0)) \in \mathcal{R}(\sigma)$  from which we get  $(M', f^{k+1}(0)) \in \mathcal{R}(\sigma)$  by Lemma 20 and this ends the proof of (3.8). We conclude that  $(M', [M]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\sigma)$  by Lemma 29.

Assume that  $M = \text{der}(N)$  with  $\mathcal{P} \vdash N : !\sigma$ . By inductive hypothesis we have  $(N', [N]^{\vec{x}}(\vec{v})) \in \mathcal{R}(!\sigma)$  which implies  $(\text{der}(N'), \text{der}([N]^{\vec{x}}(\vec{v}))) \in \mathcal{R}(\sigma)$  by Lemma 26, that is  $(M', [M]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\sigma)$ .

Assume that  $M = \text{pr}_j N$  with  $j \in \{1, 2\}$ ,  $\sigma = \varphi_1 \otimes \varphi_2$  and  $\mathcal{P} \vdash M : \varphi_1 \otimes \varphi_2$ . By inductive hypothesis we have  $(N', [N]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\varphi_1 \otimes \varphi_2)$  and hence  $(\text{pr}_j N', \text{pr}_j([N]^{\vec{x}}(\vec{v}))) \in \mathcal{R}(\varphi_j)$  by Lemma 23 that is  $(M', [M]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\varphi_j)$ .

Assume that  $M = \text{case}(N, y_1 \cdot R_1, y_2 \cdot R_2)$  with  $\mathcal{P} \vdash N : \varphi_1 \oplus \varphi_2$  and  $\mathcal{P}, y_j : \varphi_j \vdash R_j : \sigma$  for  $j = 1, 2$ . By inductive hypothesis we have  $(N', [N]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\varphi_1 \oplus \varphi_2)$  and  $(\lambda y_j^{\varphi_j} R_j, [\lambda y_j^{\varphi_j} R_j]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\varphi_j \multimap \sigma)$  for  $j = 1, 2$  (to prove this latter fact, one chooses  $\varepsilon \in \{-, +\}$  and considers an arbitrary  $(V, v) \in \mathcal{V}(\varphi_j)^{-\varepsilon}$ , we have  $(V, v) \in \mathcal{R}(\varphi_j)$  and hence  $(R'_j[V/y_j], [R_j]^{\vec{x}, y_j}(\vec{v}, v)) \in \mathcal{R}(\sigma)$  by inductive hypothesis, which implies

$$(\langle \lambda y_j^{\varphi_j} R'_j \rangle V, [\lambda y_j^{\varphi_j} R_j]^{\vec{x}}(\vec{v}))(v) \in \mathcal{R}(\sigma)$$

by Lemma 20). By Lemma 25 we get

$$(\text{case}(N', y_1 \cdot R'_1, y_2 \cdot R'_2), \text{case}([\lambda y_1^{\varphi_1} R_1]^{\vec{x}}(\vec{v}), [\lambda y_2^{\varphi_2} R_2]^{\vec{x}}(\vec{v}))([N]^{\vec{x}}(\vec{v}))) \in \mathcal{R}(\sigma)$$

that is  $(M', [M]^{\vec{x}}(\vec{v})) \in \mathcal{R}(\sigma)$ , by Lemma 20.

Assume that  $M = \text{unfold}(N)$  where  $\mathcal{P} \vdash N : \varphi$  with  $\varphi = \text{Fix } \zeta \cdot \psi$ . We apply Lemma 27 straightforwardly.

Assume that  $M = ()$  and the typing derivation consists of the axiom  $\mathcal{P} \vdash () : \mathbf{1}$  so that  $\sigma = \mathbf{1}$ . We have  $(M, [M]) \in \mathcal{R}(\mathbf{1})$  by definition since  $\text{Red}_{M, ()}^\infty = 1 = [M]$ .

Assume last that  $M = \text{coin}(p)$  for some  $p \in [0, 1] \cap \mathbb{Q}$  and the typing derivation consists of the axiom  $\mathcal{P} \vdash \text{coin}(p) : \mathbf{1} \oplus \mathbf{1}$  so that  $\sigma = \mathbf{1} \oplus \mathbf{1}$ . We must prove that  $(\text{coin}(p), [\text{coin}(p)]) \in \mathcal{R}(\mathbf{1} \oplus \mathbf{1})$ . Remember that  $[\text{coin}(p)] = pe_{(1,*)} + (1-p)e_{(2,*)}$ . Let  $\varepsilon \in \{-, +\}$  and let  $(T, t) \in (\mathcal{V}(\mathbf{1} \oplus \mathbf{1}) \multimap \mathcal{R}(\mathbf{1}))^\varepsilon$ , we must prove that  $(\langle T \rangle \text{coin}(p), t(pe_{(1,*)} + (1-p)e_{(2,*)})) \in \mathcal{R}(\mathbf{1})$ . We have

$$\text{Red}_{\langle T \rangle \text{coin}(p), ()}^\infty = p \text{Red}_{\langle T \rangle \text{in}_1(), ()}^\infty + (1-p) \text{Red}_{\langle T \rangle \text{in}_2(), ()}^\infty$$

since the first reduction step must be  $\text{coin}(p) \xrightarrow{p} \text{in}_1()$  or  $\text{coin}(p) \xrightarrow{1-p} \text{in}_2()$ . By our assumption on  $(T, t)$  we have  $\text{Red}_{\langle T \rangle \text{in}_i(), ()}^\infty \geq t(e_{(i,*)})$  and hence  $\text{Red}_{\langle T \rangle \text{coin}(p), ()}^\infty \geq t(pe_{(1,*)} + (1-p)e_{(2,*)})$  as contended, by linearity of  $t$ .  $\square$



**Theorem 37** (Adequacy). *Let  $M$  be a closed term such that  $\vdash M : \mathbf{1}$ . Then  $[M] = \text{Red}_{M, ()}^\infty$ .*

*Proof.* By Theorem 36 we have  $[M] \leq \text{Red}_{M, ()}^\infty$ . We prove the converse. By Theorem 17 we have  $[M] = \sum_{\vdash M' : \mathbf{1}} \text{Red}_{M, M'}^n [M'] \geq \sum_{\vdash M' : \mathbf{1}} \text{Red}_{M, M'}^n [M'] \geq \text{Red}_{M, ()}^n$  for each  $n \in \mathbb{N}$ . The announced inequality results from the fact that  $\text{Red}_{M, ()}^\infty = \sup_{n \in \mathbb{N}} \text{Red}_{M, ()}^n$ .  $\square$

#### 4. FULL ABSTRACTION

We prove now the converse of the Adequacy Theorem. For this purpose we associate *testing terms* with the points of the model. More precisely:

- Given a positive type  $\varphi$  and  $a \in ||\varphi||$ , we define a term  $a^0$  such that

$$\vdash a^0 : !\iota \multimap \varphi \multimap \mathbf{1}.$$

- Given a general type  $\sigma$  and  $a \in ||\sigma||$ , we define terms  $a^-$  and  $a^+$  such that

$$\vdash a^- : !\iota \multimap !\sigma \multimap \mathbf{1}$$

$$\vdash a^+ : !\iota \multimap \sigma.$$

The main observation (see Lemma 39) is that for a closed term  $M$  of type  $\sigma$ , the semantics of  $\langle a^- \rangle$  is an entire series with finitely many parameters and whose coefficient of the unitary monomial<sup>9</sup> can be seen as a morphism in  $\mathbf{P}[!\sigma \multimap \mathbf{1}]$  (see Lemma 38). Moreover, when applied to  $[M]$ , this coefficient is equal to  $\mathfrak{m}^-(a) [M]_a$  where  $\mathfrak{m}^-(a) \neq 0$  depends only on  $a$ . Therefore (see Theorem 40) two closed terms  $M_i$  with different semantics, i.e. such that there is a point  $a \in ||\sigma||$  such that  $[M_1]_a \neq [M_2]_a$ , will generate two different entire series  $[\lambda x^{!\iota} \langle \langle a^- \rangle x \rangle M_i^!]$ . Finally, it is possible to find parameters  $\vec{\zeta}$  such that  $\langle a^- \rangle \text{ran}(\vec{\zeta})^!$  separates  $M_1$  and  $M_2$  (where  $\text{ran}(\vec{\zeta})$  has been introduced in Paragraph 2.3). Indeed,  $\langle \langle a^- \rangle \text{ran}(\vec{\zeta})^! \rangle M_i^!$  is interpreted as  $[\lambda x^{!\iota} \langle \langle a^- \rangle x \rangle M_i^!](\vec{\zeta})$  and by Adequacy Theorem 37.

**4.1. Notations.** For any  $u \in \mathbf{P}[\sigma]$  and  $v \in \mathbf{P}[\varphi]$ ,  $[a^0] v$ ,  $[a^-](u)$  and  $[a^+]$  are entire series with real non-negative coefficients over  $\mathbf{P}[\iota]$ . Notice that an element of  $\mathbf{P}[\iota]$  is a potentially infinite sequence  $\vec{\zeta} = (\zeta_1, \dots, \zeta_n, \dots)$  of non-negative reals  $\zeta_i$  indexed by integers and that we call parameters. Moreover, for  $\vec{\zeta} \in \mathbf{P}[\iota]$ ,  $[a^0](\vec{\zeta})u$ ,  $[a^-](\vec{\zeta})(u)$  and  $[a^+](\vec{\zeta})$  depend on finitely many parameters corresponding respectively to the  $|a|^0$ ,  $|a|^-$  and  $|a|^+$  first coefficients of the sequence  $\vec{\zeta}$  (where  $|a|^0$ ,  $|a|^-$  and  $|a|^+$  are natural numbers depending only on  $a$ ). This means that monomials with non-zero coefficients in these power series are products of the finitely many first parameters  $(\zeta_1, \dots, \zeta_n)$ . Besides, when handling several finite sequences of parameters<sup>10</sup>  $\vec{\zeta}_1$  and  $\vec{\zeta}_2$ , we will assume that they are disjoint even if it requires a renaming. We denote as  $\mathfrak{m}^0(a)$ ,  $\mathfrak{m}^-(a)$  and  $\mathfrak{m}^+(a)$  natural numbers depending only on  $a$  and that will appear as coefficients of the corresponding entire series.

For any term  $M$  of type  $!\iota \multimap \sigma$  and  $P$  of type  $\iota$ , we use the notation  $M(P) = \langle M \rangle P^!$  which is coherent with the notation for the composition in the Kleisli Category (see Paragraph 3.2.7).

<sup>9</sup>That is, the monomial where each exponent is equal to one.

<sup>10</sup>We follow the mathematical habits of using the same notation  $\vec{\zeta} = (\zeta_1, \dots, \zeta_n)$  to refer to parameters of a formal power series and to real coefficients of the mate function.



We use the terms introduced in the probabilistic tests paragraph of Subsection 2.3 and whose semantics is given in Subsection 3.3.

**4.2. Testing terms.** We give the definition of the terms  $a^0$ ,  $a^-$  and  $a^+$  and the associated natural numbers  $|a|^0$ ,  $|a|^-$  and  $|a|^+$ , by induction on the structure of the point  $a$ .

Let  $\varphi$  be a positive type. Let us first define  $a^-$ , as it does not depend on the structure of  $\varphi$ :

$$a^- = \lambda Z^{\iota} \lambda x^{\iota\varphi} \langle \langle a^0 \rangle Z \rangle \text{der}(x), \quad \mathfrak{m}^-(a) = \mathfrak{m}^0(a) \quad \text{and} \quad |a|^- = |b|^-.$$

Now, we define  $a^0$  and  $a^-$  by induction on the size of  $a$  using the structure of  $\varphi$ .

Let  $\varphi = !\sigma$  and  $a = [b_1, \dots, b_k]$  with  $b_i \in [|\sigma|]$ . By inductive hypothesis, we have built terms  $\vdash b_i^- : !\iota \multimap !\sigma \multimap \mathbf{1}$  and  $\vdash b_i^+ : !\iota \multimap \sigma$ . Then we set

$$\begin{aligned} a^0 &= \lambda Z^{\iota} \lambda x^{\iota\sigma} \langle b_1^- (\langle \text{win}_1(|b_1|^- , \dots, |b_k|^-) \rangle \text{der}(Z)) \rangle x \wedge \dots \\ &\quad \wedge \langle b_k^- (\langle \text{win}_k(|b_1|^- , \dots, |b_k|^-) \rangle \text{der}(Z)) \rangle x, \\ \mathfrak{m}^0(a) &= \prod_{i=1}^k \mathfrak{m}^-(b_i), \quad \text{and} \quad |a|^0 = |b_1|^- + \dots + |b_k|^- . \end{aligned}$$

$$a^+ = \lambda Z^{\iota} \sum_{i=1}^k \text{der}(Z)_{\underline{i}} \cdot (b_i^+ (\langle \text{win}_i(k + |b_1|^+ , |b_2|^+ , \dots, |b_k|^+) \rangle \text{der}(Z)))^!,$$

$$\mathfrak{m}^+(a) = a! \prod_{i=1}^k \mathfrak{m}^+(b_i), \quad \text{and} \quad |a|^+ = k + |b_1|^+ + \dots + |b_k|^+.$$

Remember that the factorial  $a!$  of a multiset  $a$  has been defined in Paragraph 3.2.7 as the number of permutations that fix  $a$ .

If  $\varphi = \varphi_1 \otimes \varphi_2$  and  $a = (b_1, b_2)$  with  $b_i \in [|\varphi_i|]$  for  $i = 1, 2$ , then we set

$$\begin{aligned} a^0 &= \lambda Z^{\iota} \lambda x^{\varphi} \langle b_1^0 (\langle \text{win}_1(|b_1|^0 , |b_2|^0) \rangle \text{der}(Z)) \rangle \text{pr}_1 x \wedge \langle b_2^0 (\langle \text{win}_2(|b_1|^0 , |b_2|^0) \rangle \text{der}(Z)) \rangle \text{pr}_2 x, \\ \mathfrak{m}^0(a) &= \mathfrak{m}^0(b_1) \mathfrak{m}^0(b_2), \quad \text{and} \quad |a|^0 = |b_1|^0 + |b_2|^0 . \end{aligned}$$

$$\begin{aligned} a^+ &= \lambda Z^{\iota} (b_1^+ (\langle \text{win}_1(|b_1|^+ , |b_2|^+) \rangle \text{der}(Z)) , b_2^+ (\langle \text{win}_2(|b_1|^+ , |b_2|^+) \rangle \text{der}(Z))) , \\ \mathfrak{m}^+(a) &= \mathfrak{m}^+(b_1) \mathfrak{m}^+(b_2) \quad \text{and} \quad |a|^+ = |b_1|^+ + |b_2|^+ . \end{aligned}$$

If  $\varphi = \varphi_1 \oplus \varphi_2$  and  $a = (1, b)$  with  $b \in [|\varphi_1|]$  (the case  $a = (2, b)$  is similar), then we set

$$a^0 = \lambda Z^{\iota} \lambda x^{\varphi_1 \oplus \varphi_2} \text{case}(x, y_1 \cdot \langle \langle b^0 \rangle Z \rangle y_1, y_2 \cdot \Omega^{\mathbf{1}}), \quad \mathfrak{m}^0(a) = \mathfrak{m}^0(b) \quad \text{and} \quad |a|^0 = |b|^0.$$

$$a^+ = \lambda Z^{\iota} \text{in}_1 b^+(Z), \quad \mathfrak{m}^+(a) = \mathfrak{m}^+(b) \quad \text{and} \quad |a|^+ = |b|^+.$$

Finally, for a general type  $\sigma$  and  $a \in [|\sigma|]$ , we define  $a^-$  and  $a^+$ .

If  $\sigma = \varphi$  is positive, then we have already defined  $a^-$  and  $a^+$ .

If  $\sigma = \varphi \multimap \tau$  and  $a = (b, c)$  with  $b \in [|\varphi|]$  and  $c \in [|\tau|]$  then we set

$$\begin{aligned} a^- &= \lambda Z^{\iota} \lambda f^{\iota(\varphi \multimap \tau)} \langle c^- (\langle \text{win}_2(|b|^+ , |c|^-) \rangle \text{der}(Z)) \rangle (\langle \text{der}(f) \rangle b^+ (\langle \text{win}_1(|b|^+ , |c|^-) \rangle \text{der}(Z)))^! \\ \mathfrak{m}^-(a) &= \mathfrak{m}^+(b) \mathfrak{m}^-(c), \quad \text{and} \quad |a|^- = |b|^+ + |c|^- . \end{aligned}$$

$$a^+ = \lambda Z^{!l} \lambda x^\varphi \langle b^0 (\langle \text{win}_1(|b|^0, |c|^+) \rangle \text{der}(Z)) \rangle x \cdot c^+ (\langle \text{win}_2(|b|^0, |c|^+) \rangle \text{der}(Z)),$$

$$\mathfrak{m}^+(a) = \mathfrak{m}^0(b) \mathfrak{m}^+(c), \text{ and } |a|^+ = |b|^0 + |c|^+.$$

It is easy to check that these terms satisfy the announced typing judgments. It is also clear that  $\mathfrak{m}^0(a)$ ,  $\mathfrak{m}^-(a)$  and  $\mathfrak{m}^+(a)$  are non zero natural numbers.

Next Lemma 38 gives coefficient of the unitary monomial and provides properties which will show useful in the proof of the key Lemma 39.

**Lemma 38.** *Let  $\sigma$  be a general type and  $t \in \mathbf{P}[!l \multimap \sigma]$ . Assume that there is  $k \in \mathbb{N}$  such that for any  $c \in ||\sigma||$ , the entire series  $\widehat{t}_c$  over  $\mathbf{P}[l]$  depends on  $k$  parameters. For any  $c \in ||\sigma||$ , let us denote as  $\mathfrak{c}_\zeta^1(\widehat{t})_c$  the coefficient of the monomial  $\zeta_1 \dots \zeta_k$  of  $\widehat{t}_c$ . Then,  $k^{-k} \mathfrak{c}_\zeta^1(\widehat{t}) \in \mathbf{P}[\sigma]$ .*

*Let  $\varphi$  be a positive type,  $\tau$  a general type and  $t \in \mathbf{P}[!l \multimap (\varphi \multimap \tau)]$ . Assume that there is  $k \in \mathbb{N}$  such that for any  $(a, b) \in |\varphi \multimap \tau|$ ,  $\widehat{t}_{(a,b)}$  depends on  $k$  parameters, then  $k^{-k} \mathfrak{c}_\zeta^1(\widehat{t}) \in \mathbf{P}[\varphi \multimap \tau]$ . If there is  $m \in \mathbf{P}[\tau]$  and  $a \in ||\varphi||$  such that for all  $u \in \mathbf{P}^![\varphi]^!$ ,  $\mathfrak{c}_\zeta^1(\widehat{t}) u = m u_a$ , then the same property holds for all  $u \in \mathbf{P}[\varphi]$ .*

*Proof.* First notice that  $\forall c \in ||\sigma||$ , the coefficient of the monomial  $\prod_{i=1}^k \zeta_i$  is  $\mathfrak{c}_\zeta^1(\widehat{t})_c = t_{([1, \dots, k], c)}$ . Now, let  $\vec{\frac{1}{k}}$  be the sequence of  $k$  coefficients all equal to  $\frac{1}{k}$ :

$$\widehat{t}_c(\vec{\frac{1}{k}}) = \sum_{\substack{\mu \in ||l|| \\ \text{supp}(\mu) \subseteq \{1, \dots, k\}}} t_{(\mu, c)} \frac{1}{k}^{\#\mu}$$

so that  $\mathfrak{c}_\zeta^1(\widehat{t})_c k^{-k} \leq \widehat{t}(\vec{\frac{1}{k}})_c$ . Since  $\vec{\frac{1}{k}} \in \mathbf{P}[l]$ ,  $\widehat{t}(\vec{\frac{1}{k}}) \in \mathbf{P}[\sigma]$  which is downward closed, we have that  $k^{-k} \mathfrak{c}_\zeta^1(\widehat{t}) \in \mathbf{P}[\sigma]$ .

For any  $a \in ||\varphi||$ , there is  $A$  such that  $u_a \leq A$  for any  $u \in \mathbf{P}[\varphi]$  (see Subsection 3.2). Hence, for any  $u \in \mathbf{P}[\varphi]$ ,  $\frac{u_a}{A} m \in \mathbf{P}[\tau]$  and we deduce thanks to Lemma 3 that  $u \mapsto u_a m A^{-1}$  is in  $\mathbf{P}[\varphi \multimap \tau]$ . Without loss of generality, we can choose  $A \geq k^k$  so that  $u \mapsto u_a m A^{-1}$  and  $u \mapsto A^{-1} \mathfrak{c}_\zeta^1(\widehat{t}) u$  are both in  $\mathbf{P}[\varphi \multimap \tau]$ . Now, since  $[\varphi]$  is dense (see Theorem 16), if  $u \mapsto A^{-1} \mathfrak{c}_\zeta^1(\widehat{t}) u$  and  $u \mapsto A^{-1} u_a m$  are equal on coalgebraic point  $u \in \mathbf{P}^![\varphi]^!$ , they are equal (see Definition 12). Thus, for all  $u \in \mathbf{P}[\varphi]$ ,  $u_a m = \mathfrak{c}_\zeta^1(\widehat{t}) u$ .  $\square$

**Lemma 39.** *Let  $\sigma$  be a general type and  $a \in ||\sigma||$ .*

*For any  $a' \in ||\sigma||$ ,  $\widehat{[a^-]}_{(a', *)}$  is an entire series over  $\mathbf{P}[l]$  depending on  $|a|^-$  parameters,  $\mathfrak{c}_\zeta^1([a^-]) \in \mathbf{P}[! \sigma \multimap \mathbf{1}]$  and for any  $u \in \mathbf{P}[\sigma]$ ,  $\mathfrak{c}_\zeta^1([a^-])(u) = \mathfrak{m}^-(a) u_a$ .*

*For any  $a' \in ||\sigma||$ ,  $\widehat{[a^+]}_{a'}$  is an entire series over  $\mathbf{P}[l]$  depending on  $|a|^+$  parameters,  $\mathfrak{c}_\zeta^1([a^+]) \in \mathbf{P}[\sigma]$  and  $\mathfrak{c}_\zeta^1([a^+]) = \mathfrak{m}^+(a) e_a$ .*

*Let  $\sigma = \varphi$  be positive. If  $a' \in ||\varphi||$ , then  $\widehat{[a^0]}_{(a', *)}$  is an entire series over  $\mathbf{P}[l]$  depending on  $|a|^0$  parameters,  $\mathfrak{c}_\zeta^1([a^0]) \in \mathbf{P}[\varphi \multimap \mathbf{1}]$  and for any  $u \in \mathbf{P}([\varphi])$ ,  $\mathfrak{c}_\zeta^1([a^0]) u = \mathfrak{m}^0(a) u_a$ .*

*Proof.* Let us argue by mutual induction on the structure of  $a$ .

If  $\varphi = !\sigma$  and  $a = [b_1, \dots, b_k]$ . Let  $a' = [b'_1, \dots, b'_{k'}] \in [! \sigma]$  and  $\vec{\zeta} \in \mathbf{P}[l]$  be the concatenation of the finite sequences  $\vec{\zeta}^i \in \mathbf{P}[l]$  such that the length of  $\vec{\zeta}^0$  is  $k$  and the length

of  $\vec{\zeta}^i$  is  $|b_i|^+$  for  $i \geq 1$ .

$$[a^+] (\vec{\zeta})_{a'} = \left( \sum_{i=1}^k \zeta_i^0 [b_i^+] (\vec{\zeta}^i) \right)_{a'}^! = \prod_{j=1}^{k'} \left( \sum_{i=1}^k \zeta_i^0 [b_i^+] (\vec{\zeta}^i) \right)_{b'_j}.$$

We want to compute the coefficient of the unitary monomial which contains exactly one copy of each parameter of every  $\vec{\zeta}^i$ . If  $k' \neq k$ , then it is not possible to get a monomial with exactly once each parameter of  $\vec{\zeta}^0$ , so that  $\mathfrak{c}_{\vec{\zeta}}^1([a^+])_{a'} = 0$ . If  $k' = k$  and  $\mathfrak{S}_k$  is the set of permutations over  $k$ , then by using the fact that  $a! = \#\{\rho \in \mathfrak{S}_k \mid \forall i b_i = b_{\rho(i)}\}$ , by denoting the Kronecker symbol as  $\delta_{a,a'}$  and by the induction hypothesis, we get that:

$$\mathfrak{c}_{\vec{\zeta}}^1([a^+])_{a'} = \sum_{\rho \in \mathfrak{S}_k} \prod_{i=1}^k \mathfrak{c}_{\vec{\zeta}^i}^1([b_i^+])_{b'_{\rho(i)}} = a! \prod_{i=1}^k \mathfrak{m}^+(b_i) \delta_{b_i, b'_i} = \mathfrak{m}^+(a) \delta_{a,a'} = \mathfrak{m}^+(a) (e_a)_{a'}.$$

Let  $u \in \mathbf{P}^!([\varphi]^!)$ , that is  $u \in \mathbf{Pcoh}^!(1, [\varphi]^!)$  is coalgebraic, let  $\vec{\zeta} \in \mathbf{P}[\mathcal{L}]$  be the concatenation of the finite sequences  $\vec{\zeta}^i \in \mathbf{P}[\mathcal{L}]$  such that the length of  $\vec{\zeta}^i$  is  $|b_i|^-$ . Then

$$[a^0] (\vec{\zeta})u = \prod_{i=1}^k [b_i^-] (\vec{\zeta}^i)(u) \quad \text{and} \quad \mathfrak{c}_{\vec{\zeta}}^1([a^0]) u = \prod_{i=1}^k \mathfrak{c}_{\vec{\zeta}^i}^1([b_i^-]) (\text{der}_{[\sigma]} u).$$

Besides, since  $u$  is coalgebraic by assumption, we know that  $u = \text{der}_{[\sigma]}(u)^!$  (see Lemma 13). By inductive hypothesis it follows that

$$\mathfrak{c}_{\vec{\zeta}}^1([a^0]) u = \prod_{i=1}^k \mathfrak{m}^-(b_i) \text{der}_{[\sigma]}(u)_{b_i} = \mathfrak{m}^0(a) u_a.$$

Hence, by Lemma 38, we have  $\mathfrak{c}_{\vec{\zeta}}^1([a^0]) u = \mathfrak{m}^0(a) u_a$  for all  $u \in \mathbf{P}[\varphi]$ .

Let  $\varphi = \varphi_1 \otimes \varphi_2$  and  $a = (b_1, b_2)$  with  $a_i \in |[\varphi_i]|$ . Let  $\vec{\zeta} \in \mathbf{P}[\mathcal{L}]$  be the concatenation of the finite sequences  $\vec{\zeta}^1, \vec{\zeta}^2 \in \mathbf{P}[\mathcal{L}]$  such that the length of  $\vec{\zeta}^i$  is  $|b_i|^+$ . Let  $a' = (b'_1, b'_2) \in |[\varphi_1 \otimes \varphi_2]|$ .

$$[a^+] (\vec{\zeta}) = [b_1^+] (\vec{\zeta}^1) \otimes [b_2^+] (\vec{\zeta}^2) \quad \text{and} \quad \mathfrak{c}_{\vec{\zeta}}^1([a^+])_{a'} = \mathfrak{c}_{\vec{\zeta}^1}^1([b_1^+])_{b'_1} \mathfrak{c}_{\vec{\zeta}^2}^1([b_2^+])_{b'_2}$$

By inductive hypothesis, we get that  $\mathfrak{c}_{\vec{\zeta}}^1([a^+])_{a'} = \mathfrak{m}^+(b_1) \delta_{b_1, b'_1} \mathfrak{m}^+(b_2) \delta_{b_2, b'_2} = \mathfrak{m}^+(a) \delta_{a,a'}$ .

If  $u \in \mathbf{P}^!([\varphi]^!)$ , then  $u = u_1 \otimes u_2$  where  $u_i = \text{pr}_i^\otimes(u) \in \mathbf{P}^!([\varphi_i]^!)$  for  $i = 1, 2$  (see Lemma 13). Therefore

$$[a^0] (\vec{\zeta})u = [b_1^0] (\vec{\zeta}^1)u_1 [b_2^0] (\vec{\zeta}^2)u_2 \quad \text{and} \quad \mathfrak{c}_{\vec{\zeta}}^1([a^0]) u = \mathfrak{c}_{\vec{\zeta}^1}^1([b_1^0]) u_1 \mathfrak{c}_{\vec{\zeta}^2}^1([b_2^0]) u_2$$

and hence by inductive hypothesis  $\mathfrak{c}_{\vec{\zeta}}^1([a^0]) u = \mathfrak{m}^0(b_1)(u_1)_{b_1} \mathfrak{m}^0(b_2)(u_2)_{b_2} = \mathfrak{m}^0(a)u_a$  for  $u \in \mathbf{P}^!([\varphi]^!)$  and hence for  $u \in \mathbf{P}[\varphi]$  by Lemma 38.

If  $\varphi = \varphi_1 \oplus \varphi_2$  and  $a = (1, a_1)$  with  $a_1 \in |[\varphi_1]|$  (the case  $a = (2, a_2)$  is similar), then

$$[a^+] (\vec{\zeta}) = [\text{in}_1 a_1^+] (\vec{\zeta}) \quad \text{and} \quad \mathfrak{c}_{\vec{\zeta}}^1([a^+])_{(i,a')} = \delta_{1,i} \mathfrak{c}_{\vec{\zeta}}^1([a_1^+])_{a'} = \mathfrak{m}^+(a) \delta_{a,(i,a')}$$

Let  $u \in \mathbf{P}^!([\varphi]^!)$ . There is  $i \in \{1, 2\}$  such that  $u = \text{in}_i u_i$  with  $u_i \in \mathbf{P}^!([\varphi_i]^!)$  (see Lemma 13). If  $i = 1$ , then

$$[a^0] (\vec{\zeta})u = [\lambda x^{\varphi_1 \oplus \varphi_2} \text{case}(x, y_1 \cdot \langle a_1^0 \rangle y_1, y_2 \cdot \Omega^1)] (\vec{\zeta})(u) = [a_1^0] (\vec{\zeta})u_1$$

and if  $i = 2$  then  $[a^0](\vec{\zeta})u = [\Omega^1] = 0$ . So that  $\mathbb{C}_{\vec{\zeta}}^1([a^0])u = \mathfrak{m}^0(a_1)(u_1)_{a_1} = \mathfrak{m}^0(a)u_a$  for  $u \in \mathbf{P}^1[\varphi]^!$  and hence for  $u \in \mathbf{P}[\varphi]$  by Lemma 38.

Let now  $\varphi$  be a positive type and let  $a \in ||[\varphi]||$ . Let  $u \in \mathbf{P}[\varphi]$ , we have

$$[a^-](\vec{\zeta})(u) = [a^-](\vec{\zeta})u^! = [a^0](\vec{\zeta})u \quad \text{and} \quad \mathbb{C}_{\vec{\zeta}}^1([a^-])(u) = \mathbb{C}_{\vec{\zeta}}^1([a^0])u$$

By induction hypothesis,  $\mathbb{C}_{\vec{\zeta}}^1([a^0])u = \mathfrak{m}^0(a)u_a = \mathfrak{m}^-(a)u_a$ .

Last, let  $\sigma = \varphi \multimap \tau$  and let  $a = (b, c) \in ||[\sigma]||$ . Let  $\vec{\zeta} \in \mathbf{P}[\iota]$  be the concatenation of the finite sequences  $\vec{\zeta}^1, \vec{\zeta}^2 \in \mathbf{P}[\iota]$  such that the length of  $\vec{\zeta}^1$  is  $|b|^0$  and the length of  $\vec{\zeta}^2$  is  $|c|^+$ . Then,  $[a^+]$  is an entire series over  $\mathbf{P}[\iota]$  with values in  $\mathbf{P}[\varphi \multimap \tau]$ . By Lemma 38,  $\mathbb{C}_{\vec{\zeta}}^1([a^+])$  is in  $\mathbf{P}[\varphi \multimap \tau]$  and thus can be seen as a morphism in  $\mathbf{Pcoh}([\varphi], [\tau])$ . By inductive hypothesis, for each  $u \in \mathbf{P}[\varphi]$ ,

$$\begin{aligned} [a^+](\vec{\zeta})u &= [b^0](\vec{\zeta}^1)u \cdot [c^+](\vec{\zeta}^2) \\ \text{and } \mathbb{C}_{\vec{\zeta}}^1([a^+])u_{c'} &= \mathbb{C}_{\vec{\zeta}^1}^1([b^0])u_{\mathbb{C}_{\vec{\zeta}^2}^1([c^+]_{c'})} = \mathfrak{m}^0(b)u_b \mathfrak{m}^+(c)\delta_{c,c'} \end{aligned}$$

Let  $\varepsilon > 0$  such that  $\varepsilon e_{b'} \in \mathbf{P}[\varphi]$ . Since  $\mathbb{C}_{\vec{\zeta}}^1([a^+]) \in \mathbf{P}[\varphi \multimap \tau]$  is linear and by using induction hypothesis, we have

$$\mathbb{C}_{\vec{\zeta}}^1([a^+])_{(b',c')} = \frac{1}{\varepsilon}(\mathbb{C}_{\vec{\zeta}}^1([a^+])\varepsilon e_{b'})_{c'} = \mathfrak{m}^0(b)\delta_{b,b'}\mathfrak{m}^+(c)\delta_{c,c'} = \mathfrak{m}^+(a)\delta_{a,a'}.$$

Let now  $u \in \mathbf{P}[\varphi \multimap \tau]$  and  $\vec{\zeta} \in \mathbf{P}[\iota]$  be the concatenation of the finite sequences  $\vec{\zeta}^1, \vec{\zeta}^2 \in \mathbf{P}[\iota]$  such that the length of  $\vec{\zeta}^1$  is  $|b|^+$  and the length of  $\vec{\zeta}^2$  is  $|c|^-$ .

$$\begin{aligned} [a^-](\vec{\zeta})(u) &= [c^-](\vec{\zeta}^2)(u[b^+](\vec{\zeta}^1)) \\ \text{and } \mathbb{C}_{\vec{\zeta}}^1([a^-])(u) &= \mathbb{C}_{\vec{\zeta}^1}^1(\mathbb{C}_{\vec{\zeta}^2}^1([c^-])(u[b^+](\vec{\zeta}^1))) \end{aligned}$$

By induction hypothesis, we have  $\mathbb{C}_{\vec{\zeta}^2}^1([c^-](u[b^+](\vec{\zeta}^1))) = \mathfrak{m}^-(c)(u[b^+](\vec{\zeta}^1))_c$ . Moreover, since  $u \in \mathbf{P}[\varphi \multimap \tau]$ , seen as a morphism in  $\mathbf{Pcoh}([\varphi], [\tau])$  is linear, and there is  $\varepsilon > 0$  such that  $\varepsilon e_b \in \mathbf{P}[\varphi]$ , and by using induction hypothesis, we get that  $\mathbb{C}_{\vec{\zeta}^1}^1(u[b^+])_c = (u\mathbb{C}_{\vec{\zeta}^1}^1([b^+]))_c = (u\mathfrak{m}^+(b)e_b)_c = \mathfrak{m}^+(b)u_{(b,c)}$ . Therefore, we have  $\mathbb{C}_{\vec{\zeta}}^1([a^-])(u) = \mathfrak{m}^+(b)\mathfrak{m}^-(c)u_{(b,c)} = \mathfrak{m}^-(a)u_a$ . □

**Theorem 40** (Full Abstraction).

If  $\vdash M_1 : \sigma$  and  $\vdash M_2 : \sigma$  satisfy  $M_1 \sim M_2$  then  $[M_1] = [M_2]$ .

*Proof.* Towards a contradiction, assume that  $[M_1] \neq [M_2]$ . There is  $a \in ||[\sigma]||$  such that  $[M_1]_a \neq [M_2]_a$ . Then by Lemma 39,  $[\lambda x^! \langle \langle a^- \rangle x \rangle M_i^!]$ , for  $i \in \{1, 2\}$ , are entire series with different coefficients, namely the coefficients of the monomial  $\zeta_1 \dots \zeta_{|a|^-}$  are  $\mathfrak{m}^-(a)[M_i]_a$  for  $i \in \{1, 2\}$  as  $[\lambda x^! \langle \langle a^- \rangle x \rangle M_i^!](\vec{\zeta}) = [a^-](\vec{\zeta})([M_i])$ . There is  $\vec{\zeta} = (\zeta_1, \dots, \zeta_{|a|^-}) \in \mathbf{P}[\iota]$  with  $\zeta_i \in \mathbb{Q} \cap [0, 1]$  such that  $[a^-](\vec{\zeta})([M_1]) \neq [a^-](\vec{\zeta})([M_2])$ . Yet,  $\left[ \langle \langle a^- \rangle \text{ran}(\vec{\zeta})^! \rangle M_i^! \right] = [a^-](\vec{\zeta})([M_i])$ . By Theorem 37, we get that  $\langle \langle a^- \rangle \text{ran}(\vec{\zeta})^! \rangle M_1^!$  and  $\langle \langle a^- \rangle \text{ran}(\vec{\zeta})^! \rangle M_2^!$  converge to  $()$  with different probabilities. It follows that  $M_1 \not\sim M_2$ . □

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